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An Analysis of the Distribution of Numbers of the Form alpha[superscript x] beta[superscript y]

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AN ANALYSIS OF THE
DISTRIBUTION OF NUMBERS OF THE FORM $a^x b^y$

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This thesis will provide an analysis of the sequences formed in an interspersion. It is known that interspersions can be generated entirely from two infinite sequences of positive integers. Since these sequences are based on the integer parts of multiples of an irrational number, finding an expression for one is equivalent to finding an expression for the other. We shall see that sums involving the floor of an integer multiple of an irrational number θ can be evaluated by first expressing the indices of a summation in the numeration system based on the denominators of the convergents of θ.

We shall then turn our attention towards counting the number of times a particular element appears in some sequence G. It will be shown that a minimum of two repetition lengths occur.

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Interspersions

This thesis will provide an analysis of the implications of Brown's Decomposition in regard to the distribution of numbers of the form $a^x b^y$ where a and b are multiplicatively independent. Recall that two positive integers are multiplicatively independent provided $x = y = 0$ whenever $a^x = b^y$. We will use $\lfloor x \rfloor$ to denote the greatest integer that is less than or equal to x. Similarly, $\lceil x \rceil$ will represent the smallest integer that is greater than or equal to x. Finally, for $x \in \mathbb{R}$, the fractional part of x is denoted by $\{x\}$ and is defined by:

$$\{x\} = \begin{cases} x - \lfloor x \rfloor, & x \geq 0 \\ x - \lceil x \rceil, & x < 0 \end{cases} .$$

A useful construction called an interspersion developed by (Kimberling and Brown 2004) generates an infinite array of positive integers which we denote by

$$R = \{r(x, y) \mid x, y \in \mathbb{N} \cup \{0\}\}.$$

The set R will be ordered based on the usual ordering of the following subset of real numbers:

$$\{i + j\theta \mid i, j \in \mathbb{N} \cup \{0\}\} .$$

Throughout this thesis, we shall assume that θ is an irrational number, with $0 < \theta < 1$. Then it can be shown that the rank of each element; that is, the order of occurrence when all elements are placed in ascending order, is given by the expression

$$R(x, y) = \sum_{i=1}^{x} \lfloor i\theta \rfloor + \sum_{j=1}^{y} \left\lfloor j\frac{1}{\theta} \right\rfloor + x + y + xy + 1.$$

Row zero of this array we denote by

$$r_\theta(n) = n + 1 + \sum_{i=1}^{n} \lfloor i\theta \rfloor .$$

In a similar way, we denote column zero of the array by

$$c_\theta(n) = n + 1 + \sum_{j=1}^{n} \left\lfloor \frac{j}{\theta} \right\rfloor .$$

Every positive integer will occur somewhere in the array and for this construction, we use

$$\theta = \frac{\ln(a)}{\ln(b)}.$$

If we take as an example

$$\theta = \frac{\ln(2)}{\ln(3)} = 0.6309\cdots,$$

then the rank array will begin as indicated in Table 1.

Table 1: Interspersion

27	33	40	47	55	64
19	24	30	36	43	51
12	16	21	26	32	39
7	10	14	18	23	29
3	5	8	11	15	20
1	2	4	6	9	13

The rows and columns are indexed by the nonnegative integers so that element 23 has position $(4, 2)$, and

$$\begin{aligned} 23 &= r_\theta(4) + c_\theta(2) + 4 \cdot 2 - 1 \\ &= 9 + 7 + 4 \cdot 2 - 1. \end{aligned}$$

We remark that the elements of the table can be computed by applying the formula

$$r(x, y) = r_\theta(x) + c_\theta(y) + xy - 1.$$

Notation

We will use the standard notation for continued fractions. Let $\theta \in (0,1)$ be irrational. The simple continued fraction expansion of θ shall be denoted by

$$\theta = [a_0,\, a_1,\, a_2,\, \ldots] = a_0 + \cfrac{1}{a_1 + \frac{1}{a_2 + \cdots}}.$$

The truncated expression $[a_0, a_1, a_2, ..., a_n]$ yields the nth convergent to θ,

$$\frac{p_n}{q_n} = [a_0,\, a_1,\, a_2,\, \ldots, a_n].$$

The a_i's are called partial quotients and we further assume that p_n and q_n are relatively prime; i.e., $(p_n, q_n) = 1$. Here, as is customary, (a, b) denotes the greatest common divisor of a and b. Given the restriction on θ, it follows that $a_0 = 0$. The set of nonnegative integers will be denoted by $\mathbb{N}_0 = \{0,\, 1\,, 2\,, \ldots\}$. We denote the nth entry in the characteristic sequence of θ by

$$f_\theta(n) = \lfloor (n+1)\theta \rfloor - \lfloor n\theta \rfloor\,.$$

For $0 < \theta < 1$, the sequence f_θ consists of only 0's and 1's and is commonly known as a Sturmian sequence. Section 1 will explore the basic properties of Sturmian sequences and their intimate connection with interspersions.

Basic Properties of Continued Fractions

The convergents generated by truncating the continued fraction expansion of an irrational real number θ provide the best rational approximations to θ in the sense that a better approximation cannot be obtained without using a fraction with a larger denominator. The convergents p_n/q_n are defined recursively by

$$p_{n+1} = a_{n+1}p_n + p_{n-1}$$

and

$$q_{n+1} = a_{n+1}q_n + q_{n-1},$$

where $p_{-2} = 0$, $p_{-1} = 1$ and $q_{-2} = 1$, $q_{-1} = 0$. Every convergent p_n/q_n satisfies

$$\left|\theta - \frac{p_n}{q_n}\right| < \frac{1}{q_n q_{n+1}} < \frac{1}{q_n^2}. \tag{1}$$

Additionally, the sequence of denominators of convergents obey

$$q_k \geq 2^{\frac{k-1}{2}}$$

for $k \geq 2$. In fact, this inequality can be tightened since

$$\phi = \frac{\sqrt{5}-1}{2} = [0, 1, 1, \ldots]$$

has convergents that grow as slowly as possible given that every $a_i > 0$ for $i > 0$. Thus, $q_k \geq F_k$, where F_k is the kth Fibonacci number defined recursively by

$$F_{k+1} = F_k + F_{k-1},$$

where $F_1 = 1$ and $F_2 = 1$. Observe that every odd-indexed convergent is larger than θ and every even-indexed convergent is less than θ. Therefore, it follows that

$$\begin{aligned} \frac{p_n}{q_n} - \frac{1}{q_n q_{n+1}} &< \theta < \frac{p_n}{q_n} \\ -\frac{1}{q_{n+1}} &< q_n\theta - p_n < 0 \\ p_n - \frac{1}{q_{n+1}} &< q_n\theta < p_n \end{aligned} \tag{2}$$

when n is odd. When n is even,

$$\begin{aligned} \frac{p_n}{q_n} &< \theta < \frac{p_n}{q_n} + \frac{1}{q_n q_{n+1}} \\ 0 &< q_n\theta - p_n < \frac{1}{q_{n+1}} \\ p_n &< q_n\theta < p_n + \frac{1}{q_{n+1}}. \end{aligned} \tag{3}$$

Moreover the following chains of inequalities hold:

$$\frac{p_1}{q_1} > \frac{p_3}{q_3} > \cdots > \frac{p_{2k+1}}{q_{2k+1}} > \cdots$$

and

$$\frac{p_2}{q_2} < \frac{p_4}{q_4} < \cdots < \frac{p_{2k}}{q_{2k}} < \cdots$$

along with

$$\left|\theta - \frac{p_1}{q_1}\right| > \left|\theta - \frac{p_2}{q_2}\right| > \cdots > \left|\theta - \frac{p_k}{q_k}\right| > \cdots . \tag{4}$$

The Ostrowski Numeration System

The set of denominators of the convergents of θ can be used to create a numeration system. This numeration system allows every natural number to be expressed as a sum of multiples of the denominators of the convergents of θ. If we let $n \in \mathbb{N}_0$ then n can be represented uniquely by the expression

$$n = \sum_{0 \le i \le j} b_i q_i$$

where the b_i are integers that satisfy the following conditions:

$$\begin{aligned} 0 &\le b_0 < a_1, \\ 0 &\le b_i \le a_{i+1} \text{ for } i \ge 1, \\ \text{For } i &\ge 1, \text{if } b_i = a_{i+1} \text{ then } b_{i-1} = 0. \end{aligned}$$

Using the set of denominators of convergents of $\ln(2)/\ln(3)$, which has elements $\{1, 2, 3, 8, 19, \ldots\}$, as an example, the number 26 has representation

$$26 = 1 \cdot 19 + 0 \cdot 8 + 2 \cdot 3 + 0 \cdot 2 + 1 \cdot 1.$$

We direct the reader to (Allouche and Shallit 2003, 106) for a proof of this result.

Sturmian Sequences

Much work has been done in the area pertaining to the set of Sturmian sequences, of which the characteristic sequence of a real number is a subset. We can take as a standard example the characteristic sequence of

$$\phi = \frac{\sqrt{5} - 1}{2},$$

which is

$$f_\phi = 1011010110 \cdots .$$

We will see in Brown's Decomposition below that these sequences can be constructed by utilizing the continued fraction expansion of θ. There are several equivalent definitions of Sturmian sequences. We shall use the definition which says that every characteristic sequence of irrational number θ, where $0 < \theta < 1$, is a Sturmian sequence; and, every Sturmian sequence corresponds to precisely some θ, given the same restrictions on θ.

Elementary Results

Every Sturmian sequence is defined over an alphabet which is a set of symbols, typically denoted by

$$\sum_k = \{0,\, 1,\, \ldots, k-1\},$$

where $k \geq 2$. Using elements from the alphabet, words can be constructed by the operation of concatenation. As an example, consider again the golden ratio

$$\phi = \frac{\sqrt{5}-1}{2}.$$

Here we see that the Sturmian sequence $f_\phi = 1011010110\cdots$ is a word over the alphabet $\sum_2$. It is clear that f_θ is a word over $\sum_2$ for $0 < \theta < 1$. Given an infinite word w, we define the subword complexity function $p(n)$ to be the number of distinct subwords of length n appearing in w. Using f_ϕ as an example, it is easy to see that the set of subwords of length two is $\{10, 01, 11\}$ while the set of subwords of length three is $\{101, 011, 110, 010\}$. In each case, we note that

$$p(n) = n + 1.$$

This turns out to be true for every $n \geq 2$ and is one of the key characteristics of a Sturmian sequence. Notice that every subword of equal length in the example above seems to have almost the same quantity of 1's appearing in every subword. Given a subword u, we denote the number of 0's appearing in the word by $|u|_0$ and the number of 1's by $|u|_1$. It follows that

$$|u|_0 + |u|_1 = |u|\,,$$

where $|u|$ denotes the length of the subword u. A word w is said to be balanced provided that $||u|_1 - |v|_1| \leq 1$ for all subwords u and v of w satisfying the equation $|u| = |v|$. Sturmian sequences can thus be characterized as balanced, ultimately nonperiodic (nonrepeating), and infinite words over a two letter alphabet. The interested reader is referred to (Allouche and Shallit 2003, 312) for an excellent introduction to Sturmian sequences.

Brown's Decomposition

The characteristic sequence f_θ has a very nice construction method using the continued fraction expansion of θ. Define the words $X_0 = 0$, and $X_1 = 0^{a_1-1}1$. Now recursively define

$$X_n = X_{n-1}^{a_n} X_{n-2},$$

where $n \geq 2$ and

$$X_{n-1}^{a_n} = \underbrace{X_{n-1} \cdots X_{n-1}}_{a_n \text{ times}}$$

is a formal string of 0's and 1's created by juxtaposing the strings X_{n-1}. Interestingly, X_n is precisely the prefix of f_θ of length q_n. The prefix X_n is sometimes referred to as the nth characteristic block of θ. This result can be made to utilize the Ostrowski numeration system. Let $b_s b_{s-1} \cdots b_0$ be the Ostrowski θ representation of a positive integer m. Then,

$$f_\theta(1) f_\theta(2) \cdots f_\theta(m) = X_s^{b_s} X_{s-1}^{b_{s-1}} \cdots X_0^{b_0}.$$

If we take as an example

$$\theta = \frac{\ln(2)}{\ln(3)} = [0, 1, 1, 1, 2, \cdots],$$

then

$$X_2 = X_1^1 X_0 = 10.$$

Continuing,

$$X_3 = X_2^1 X_1 = 101.$$

2 EVALUATING SUMS OF THE FORM $\sum_{i=1}^{n} \lfloor i\theta \rfloor$

Basic Results

The first problem we wish to solve is finding an explicit formula for the quantity of numbers of the form $a^i b^j$ that lie in the half open interval $(a^x b^y, a^{x+q} b^{y-p}]$. From the definition of the rank of each element in the rank array, the value of

$$R(x,y) - R(x+q, y-p) = px - qy + p - q + pq - \sum_{i=x+1}^{x+q} \lfloor i\theta \rfloor + \sum_{j=y-p+1}^{y} \left\lfloor j\frac{1}{\theta} \right\rfloor$$

will count the quantity of numbers of the form $a^i b^j$ that lie in the interval $(a^x b^y, a^{x+q} b^{y-p}]$. For notational convenience, we define $G(x,y,p,q)$ by:

$$G(x,y,p,q) = R(x,y) - R(x+q, y-p).$$

When it is clear that the values p and q are constant, we abbreviate the former expression simply by $G(x)$. Supposing now that we hold the variable y fixed at some value and consider the differences strictly as a function of x, we will see that an expression for this can be written using only the continued fraction representation of θ. Towards this end, we determine a formula for expressions of the form

$$\sum_{i=x+1}^{x+q} \lfloor i\theta \rfloor .$$

The following three lemmas are due to (Brown and Shiue 1995, 183).

Lemma 1 *Let p_n/q_n be the nth convergent to θ. Then*

$$\lfloor q_n\theta \rfloor = \begin{cases} p_n & ,\text{for } n \text{ even} \\ p_n - 1 & ,\text{for } n \text{ odd} \end{cases} .$$

Proof. It is well known that

$$\left| \theta - \frac{p_n}{q_n} \right| < \frac{1}{q_n q_{n+1}} < \frac{1}{q_n^2}$$

We shall make frequent use of this fact. From the theory of continued fractions, we have $q_n\theta - p_n < 0$ for n odd and $q_n\theta - p_n > 0$ for n even. We assume first that n is even. Then

$$\theta - \frac{p_n}{q_n} < \frac{1}{q_n^2},$$

and

$$0 < \theta - \frac{p_n}{q_n},$$

which, when combined, yields

$$\begin{aligned} 0 &< q_n\theta - p_n < \frac{1}{q_n} \\ p_n &< q_n\theta < p_n + \frac{1}{q_n} \\ p_n &\leq q_n\theta < p_n + 1. \end{aligned}$$

Consequently it follows that

$$\lfloor q_n\theta \rfloor = p_n.$$

Similarly, for n odd,

$$\theta - \frac{p_n}{q_n} > -\frac{1}{q_n^2}$$

and

$$\theta - \frac{p_n}{q_n} < 0,$$

which, when combined, yields

$$\begin{aligned} -\frac{1}{q_n} &< q_n\theta - p_n < 0 \\ p_n - \frac{1}{q_n} &< q_n\theta < p_n \\ p_n - 1 &\leq q_n\theta < p_n \end{aligned}$$

Hence, we have

$$\lfloor q_n\theta \rfloor = p_n - 1.$$

This completes the proof. ■

Lemma 2 *If* $1 \leq j < q_n$, *then*

$$\lfloor j\theta \rfloor = \left\lfloor j\frac{p_n}{q_n} \right\rfloor.$$

Proof. Suppose that n is even. Then we have

$$\begin{aligned} 0 &< j(\theta - \frac{p_n}{q_n}) < j\frac{1}{q_n q_{n+1}} \\ j\frac{p_n}{q_n} &< j\theta < j\frac{p_n}{q_n} + j\frac{1}{q_n q_{n+1}}. \end{aligned}$$

Assuming the weakest upper bound for j, that is, $j = q_n - 1$, observe that

$$\begin{aligned} j\frac{p_n}{q_n} &< j\theta < j\frac{p_n}{q_n} + \frac{1}{q_{n+1}} - \frac{1}{q_n q_{n+1}} \\ j\frac{p_n}{q_n} &< j\theta < j\frac{p_n}{q_n} + \frac{1}{q_{n+1}} \\ \left\lfloor j\frac{p_n}{q_n} \right\rfloor + \left\{ j\frac{p_n}{q_n} \right\} &< j\theta < \left\lfloor j\frac{p_n}{q_n} \right\rfloor + \left\{ j\frac{p_n}{q_n} \right\} + \frac{1}{q_{n+1}}. \end{aligned}$$

It now is sufficient to show that $\left\{ j\frac{p_n}{q_n} \right\} + \frac{1}{q_{n+1}} < 1$. Consider the inequality

$$1 - \frac{1}{q_n} < 1 - \frac{1}{q_{n+1}}.$$

Since

$$\max_{1 \leq j < q_n} \left\{ j\frac{p_n}{q_n} \right\} = \frac{q_n - 1}{q_n} = 1 - \frac{1}{q_n},$$

it follows that

$$\left\{ j\frac{p_n}{q_n} \right\} \leq 1 - \frac{1}{q_n} < 1 - \frac{1}{q_{n+1}}.$$

From this last inequality, we obtain the inequality

$$\left\{ j\frac{p_n}{q_n} \right\} + \frac{1}{q_{n+1}} < 1.$$

Therefore,

$$\left\lfloor j\frac{p_n}{q_n}\right\rfloor + \left\{j\frac{p_n}{q_n}\right\} < j\theta < \left\lfloor j\frac{p_n}{q_n}\right\rfloor + 1.$$

Hence, the following inequality obtains:

$$\left\lfloor j\frac{p_n}{q_n}\right\rfloor \leq j\theta < \left\lfloor j\frac{p_n}{q_n}\right\rfloor + 1,$$

which in turn shows that

$$\lfloor j\theta \rfloor = \left\lfloor j\frac{p_n}{q_n}\right\rfloor.$$

For n odd,

$$\begin{aligned} j\frac{-1}{q_nq_{n+1}} &< j(\theta - \frac{p_n}{q_n}) < 0 \\ j\frac{p_n}{q_n} - j\frac{1}{q_nq_{n+1}} &< j\theta < j\frac{p_n}{q_n}. \end{aligned}$$

Again taking $j = q_n - 1$, the strongest lower bound,

$$\begin{aligned} j\frac{p_n}{q_n} - \frac{1}{q_{n+1}} + \frac{1}{q_nq_{n+1}} &< j\theta < j\frac{p_n}{q_n} \\ j\frac{p_n}{q_n} - \frac{1}{q_{n+1}} &< j\theta < j\frac{p_n}{q_n} \\ \left\lfloor j\frac{p_n}{q_n}\right\rfloor + \left\{j\frac{p_n}{q_n}\right\} - \frac{1}{q_{n+1}} &< j\theta < \left\lfloor j\frac{p_n}{q_n}\right\rfloor + \left\{j\frac{p_n}{q_n}\right\}. \end{aligned}$$

It is sufficient now to show that $\left\{j\frac{p_n}{q_n}\right\} - \frac{1}{q_{n+1}} > 0$.Since

$$\min_{1\leq j<q_n} \left\{j\frac{p_n}{q_n}\right\} = \frac{1}{q_n},$$

it follows that

$$\left\{j\frac{p_n}{q_n}\right\} \geq \frac{1}{q_n} > \frac{1}{q_{n+1}}.$$

Therefore,

$$\left\{j\frac{p_n}{q_n}\right\} - \frac{1}{q_{n+1}} > 0.$$

Moreover, we have

$$\begin{aligned}
\left\lfloor j\frac{p_n}{q_n} \right\rfloor &\leq j\theta < \left\lfloor j\frac{p_n}{q_n} \right\rfloor + \left\{ j\frac{p_n}{q_n} \right\} \\
\left\lfloor j\frac{p_n}{q_n} \right\rfloor &\leq j\theta < \left\lfloor j\frac{p_n}{q_n} \right\rfloor + 1.
\end{aligned}$$

Consequently,

$$\lfloor j\theta \rfloor = \left\lfloor j\frac{p_n}{q_n} \right\rfloor.$$

This completes the proof. ■

Lemma 3 *Let θ be a positive irrational number. Then*

$$\sum_{i=1}^{q_n} \lfloor i\theta \rfloor = \frac{1}{2}\left[p_n q_n - q_n + p_n + (-1)^n\right],$$

where p_n/q_n is the nth convergent to θ.

Proof. Consider the sum

$$\sum_{j=1}^{q_n-1} \left\{ j\frac{p_n}{q_n} \right\}.$$

Since $(p_n, q_n) = 1$, it follows that $jp_n \bmod q_n$ will take on every value from 1, to $q_n - 1$. Given that $\left\{ j\frac{p_n}{q_n} \right\} < 1$ we can write

$$\sum_{j=1}^{q_n-1} j\frac{1}{q_n} = \frac{q_n - 1}{2},$$

which is just the sum of an arithmetic series. It follows that

$$\begin{aligned}
\sum_{j=1}^{q_n-1} \left\lfloor j\frac{p_n}{q_n} \right\rfloor &= \sum_{i=1}^{q_n-1} \left(i\frac{p_n}{q_n} - \left\{ i\frac{p_n}{q_n} \right\} \right) \\
&= \frac{p_n}{q_n}\frac{(q_n-1)q_n}{2} - \frac{q_n - 1}{2} \\
&= \frac{(p_n-1)(q_n-1)}{2}.
\end{aligned}$$

By applying Lemma 1 and Lemma 2, we arrive at the result. ■

Corollary 1 *Let θ be a positive irrational. Then,*

$$\sum_{i=1}^{p_n} \left\lfloor i\frac{1}{\theta} \right\rfloor = \frac{1}{2}\left[p_n q_n + q_n - p_n - (-1)^n\right].$$

Proof. We first need the idea that if p_n/q_n is a convergent of θ and $p_n/q_n < \theta$, then p_n/q_n is a convergent of $1/\theta$ with $p_n/q_n > 1/\theta$. Then, by a similar argument,

$$\sum_{j=1}^{p_n-1} \left\lfloor j\frac{q_n}{p_n} \right\rfloor = \frac{(p_n-1)(q_n-1)}{2}.$$

The reason for the sign change will then become apparent when applying Lemma 1 and Lemma 2. ■

Evaluation of $G(x,1,1,1)$

We are now ready to express the quantity of numbers of the form $a^i b^j$ that lie in the half open interval $(a^x b, a^{x+1}]$. Notice that we have assumed that θ has $1/1$ as a convergent. This will always be the case if θ satisfies $1/2 < \theta < 1$. The value of $G(x,1,1,1)$ is given by

$$\begin{aligned} G(x,1,1,1) &= x - \sum_{i=x+1}^{x+1} \lfloor i\theta \rfloor + \sum_{j=1}^{1} \left\lfloor \frac{j}{\theta} \right\rfloor \\ &= x - \lfloor (x+1)\theta \rfloor + \left\lfloor \frac{1}{\theta} \right\rfloor. \end{aligned}$$

If we consider the consecutive differences of G, in the first variable, it follows that

$$\begin{aligned} G(x+1,1,1,1) - G(x,1,1,1) &= (x+1) - \lfloor (x+2)\theta \rfloor + \left\lfloor \frac{1}{\theta} \right\rfloor - x + \lfloor (x+1)\theta \rfloor - \left\lfloor \frac{1}{\theta} \right\rfloor \\ &= 1 - \lfloor (x+2)\theta \rfloor + \lfloor (x+1)\theta \rfloor \\ &= 1 - f_\theta(x+1). \end{aligned}$$

We now need the following simple lemma.

Lemma 4 *Let θ be irrational number with $0 < \theta < 1$. Then*

$$f_{1-\theta}(x) = 1 - f_\theta(x).$$

Proof. By direct computation, we obtain

$$\begin{aligned}
f_{1-\theta}(x) &= \lfloor (x+1)(1-\theta) \rfloor - \lfloor x(1-\theta) \rfloor \\
&= \lfloor x + 1 - x\theta - \theta \rfloor - \lfloor x - x\theta \rfloor \\
&= x + 1 - x + \lfloor -(x+1)\theta \rfloor - \lfloor -x\theta \rfloor \\
&= 1 - (\lfloor (x+1)\theta \rfloor - \lfloor x\theta \rfloor) \\
&= 1 - f_\theta(x)
\end{aligned}$$

as desired. ■

Remark 1 *This last result can be used to prove what is known as Beatty's Theorem. Accumulating each sequence will generate a pair of Beatty sequences that tile the positive integers. We refer the interested reader to (O'Bryant 2003) for a more in-depth analysis.*

We can now use Lemma 4 to find an expression for the problem above. For Theorem 1 below, let $a, b \in \mathbb{N}$ and define $\theta = \ln(a)/\ln(b)$. Further assume that a and b are such that $1/2 < \theta < 1$. We now present our first theorem.

Theorem 1 *The quantity of numbers of the form $a^i b^j$ contained in the half open interval $(a^{x+1}, a^x b]$ is*

$$\lfloor (x+1)(1-\theta) \rfloor + \left\lfloor \frac{1}{\theta} \right\rfloor.$$

Proof. Recall that by assumption, θ satisfies $0 < \theta < 1$. Thus, $\lfloor\theta\rfloor = 0$. Therefore, we have:

$$\begin{aligned}
G(x,1,1,1) &= x - \lfloor (x+1)\theta \rfloor + \left\lfloor \frac{1}{\theta} \right\rfloor \\
&= \left\lfloor \frac{1}{\theta} \right\rfloor + x - \sum_{i=1}^{x} f_\theta(i) \\
&= \left\lfloor \frac{1}{\theta} \right\rfloor + x - \sum_{i=1}^{x} (1 - f_{1-\theta}(i)) \\
&= \left\lfloor \frac{1}{\theta} \right\rfloor + x - x + \sum_{i=1}^{x} f_{1-\theta}(i) \\
&= \left\lfloor \frac{1}{\theta} \right\rfloor + \lfloor (x+1)(1-\theta) \rfloor .
\end{aligned}$$

This completes the proof. ■

Results for Several Terms

Lemma 5 *Let i and j both be odd with $i < j$. Then*

$$\lfloor q_i\theta + q_j\theta \rfloor = \lfloor q_i\theta \rfloor + \lfloor q_j\theta \rfloor + 1.$$

Proof. It suffices to show that

$$1 < \{q_i\theta\} + \{q_j\theta\} < 2.$$

From Inequality 3

$$p_i + p_j - \frac{1}{q_{i+1}} - \frac{1}{q_{j+1}} < (q_i + q_j)\theta < p_i + p_j.$$

For our result to hold, it is necessary that

$$-1 < -\left(\frac{1}{q_{i+1}} + \frac{1}{q_{j+1}}\right) < 0.$$

To see that this inequality holds, recall that for $k \geq 1$ we have $q_k \geq F_k$.Therefore, the previous inequality follows from:

$$-1 < -\left(\frac{1}{2} + \frac{1}{3}\right) \leq -\left(\frac{1}{q_2} + \frac{1}{q_4}\right) \leq -\frac{1}{q_i} - \frac{1}{q_j} < 0$$

and observing that the minimum value of $-1/q_i - 1/q_j$ occurs for $i = 1$ and $j = 3$. This completes the proof. ■

Lemma 6 *Let i and j both be even with $i < j$. Then*

$$\lfloor q_i\theta + q_j\theta \rfloor = \lfloor q_i\theta \rfloor + \lfloor q_j\theta \rfloor$$

Proof. To prove the lemma, it suffices to show that the following inequality holds:

$$\frac{1}{q_{i+1}} + \frac{1}{q_{j+1}} < 1.$$

Similarly, as in the previous lemma, observe that

$$\frac{1}{q_i} + \frac{1}{q_j} \le \frac{1}{q_3} + \frac{1}{q_5} \le \frac{1}{2} + \frac{1}{4} < 1.$$

This completes the proof. ■

Lemma 7 *Let i and j be even and odd, respectively. Then*

$$\lfloor q_i\theta + q_j\theta \rfloor = \begin{cases} \lfloor q_i\theta \rfloor + \lfloor q_j\theta \rfloor & \text{, for } i > j \\ \lfloor q_i\theta \rfloor + \lfloor q_j\theta \rfloor + 1 & \text{, for } i < j \end{cases} .$$

Proof. Consider the first case when $i > j$. Because i is even, $p_i/q_i < \theta$. Therefore, there exists a real number ϵ_i,with $0 < \epsilon_i < 1$ such that $q_i\theta = p_i + \epsilon_i$. Similarly, because j is odd $p_j/q_j > \theta$. Thus, there exists a real number ϵ_j with $0 < \epsilon_j < 1$ such that $q_j\theta = p_j - \epsilon_j$. By the decreasing nature of the odd convergents and the increasing nature of the even convergents, it follows from Inequality 4 that $\epsilon_i < \epsilon_j$. Adding the two previous equations yields

$$q_i\theta + q_j\theta = p_i + p_j + \epsilon_i - \epsilon_j.$$

Observe that the following inequality is satisfied

$$-1 < \epsilon_i - \epsilon_j < 0$$

since

$$\epsilon_i < \frac{1}{q_{i+1}},$$

and

$$\epsilon_j < \frac{1}{q_{j+1}}$$

are both true. Thus,

$$\lfloor q_i\theta + q_j\theta \rfloor = p_i + p_j - 1.$$

Applying Lemma 1 yields

$$\lfloor q_i\theta + q_j\theta \rfloor = \lfloor q_i\theta \rfloor + \lfloor q_j\theta \rfloor$$

as desired. Similarly for the case when $i < j$, it is true that

$$q_i\theta + q_j\theta = p_i + p_j + \epsilon_i - \epsilon_j.$$

As above, observe that the inequality is satisfied

$$0 < \epsilon_i - \epsilon_j < 1$$

since

$$\epsilon_i < \frac{1}{q_{i+1}},$$

and

$$\epsilon_j < \frac{1}{q_{j+1}}$$

are both true. Therefore,

$$\lfloor q_i\theta + q_j\theta \rfloor = p_i + p_j.$$

Another application of Lemma 1 yields,

$$\lfloor q_i\theta + q_j\theta \rfloor = \lfloor q_i\theta \rfloor + \lfloor q_j\theta \rfloor + 1.$$

This completes the proof. ■

We can use the previous lemmas to evaluate $G(x+1) - G(x)$ for some specific values of x. Please refer to Section 3 for those results. After completing the last three proofs, it becomes evident that the argument can be generalized. Let $k \in \mathbb{N}$. Applying the same argument given above to the following sum,

$$\sum_{n=1}^{k} q_n\theta$$

we get

$$\sum_{n=1}^{k} q_n\theta - \sum_{n=1}^{k} p_n = \sum_{n=1}^{k} (-1)^k \epsilon_n.$$

By Inequality 4, the following chain of inequalities

$$\epsilon_1 < \epsilon_2 < \cdots < \epsilon_k$$

is satisfied. Moreover, the sum of the series must satisfy

$$-1 < \sum_{n=1}^{k} (-1)^k \epsilon_n < 0$$

because the first term ϵ_1 obeys $-1 < -\epsilon_1 < 0$. Therefore, we can write

$$\begin{aligned} -1 &< \sum_{n=1}^{k} (q_n\theta) - \sum_{n=1}^{k} p_n < 0 \\ -1 + \sum_{n=1}^{k} p_n &< \sum_{n=1}^{k} q_n\theta < \sum_{n=1}^{k} p_n \\ -1 + \sum_{n=1}^{k} p_n &\leq \sum_{n=1}^{k} q_n\theta < \sum_{n=1}^{k} p_n. \end{aligned}$$

The last inequality implies that

$$\left\lfloor \sum_{n=1}^{k} q_n\theta \right\rfloor = -1 + \sum_{n=1}^{k} p_n.$$

Using this previous observation, we can now prove the following interesting result.

Proposition 1 *The following identity holds*

$$\sum_{n=1}^{k} \lfloor q_n\theta \rfloor = \left\lfloor \sum_{n=1}^{k} q_n\theta \right\rfloor - \left\lceil \frac{k}{2} \right\rceil + 1.$$

Proof. Apply the observation above in conjunction with Lemma 1. ■

The previous argument also holds if we allow only even or odd convergents. This can be seen from the fact that $q_k \geq 2^{\frac{k-1}{2}}$ for $k \geq 2$. Let us now focus on evaluating expressions of a certain form. The result of Lemma 8 and also appears in (Brown and Shiue 1995, 183).

Lemma 8 *Let $m \in \mathbb{N}$ with $q_n < m < q_{n+1}$. Write $m = jq_n + k$, where $1 \leq k < q_n$. Then*

$$\lfloor (jq_n + k)\theta \rfloor = jp_n + \lfloor k\theta \rfloor .$$

Proof. First let j be even and consider the weakest upper bound on m, that is, $m = jq_n + k = q_{n+1} - 1$. Now, multiply Inequality 3 by $jq_n + k$ to obtain

$$\begin{aligned}
jp_n + \frac{kp_n}{q_n} &< jq_n\theta + k\theta < jp_n + \frac{kp_n}{q_n} + \frac{jq_n + k}{q_nq_{n+1}} \\
jp_n + \frac{kp_n}{q_n} &< jq_n\theta + k\theta < jp_n + \frac{kp_n}{q_n} + \frac{1}{q_n} - \frac{1}{q_nq_{n+1}} \\
jp_n + \left\lfloor \frac{kp_n}{q_n} \right\rfloor + \left\{ \frac{kp_n}{q_n} \right\} &< jq_n\theta + k\theta < jp_n + \left\lfloor \frac{kp_n}{q_n} \right\rfloor + \left\{ \frac{kp_n}{q_n} \right\} + \frac{1}{q_n} - \frac{1}{q_nq_{n+1}}.
\end{aligned}$$

Because

$$0 < \left\{ \frac{kp_n}{q_n} \right\} + \frac{1}{q_n} - \frac{1}{q_nq_{n+1}} < 1$$

which follows from

$$\max_{1 \leq k < q_n} \left\{ \frac{kp_n}{q_n} \right\} = 1 - \frac{1}{q_n},$$

we can write

$$0 < \left\{ \frac{kp_n}{q_n} \right\} + \frac{1}{q_n} - \frac{1}{q_nq_{n+1}} \leq 1 - \frac{1}{q_nq_{n+1}} < 1.$$

Therefore, we have

$$jp_n + \left\lfloor \frac{kp_n}{q_n} \right\rfloor \leq jq_n\theta + k\theta < jp_n + \left\lfloor \frac{kp_n}{q_n} \right\rfloor + 1.$$

Hence, by applying Lemma 2, it follows that

$$\lfloor (jq_n + k)\theta \rfloor = jp_n + \left\lfloor \frac{kp_n}{q_n} \right\rfloor = jp_n + \lfloor k\theta \rfloor .$$

Suppose now that j is odd. Multiply Inequality 2 by $jq_n + k$ to obtain

$$\begin{aligned}
jp_n + \frac{kp_n}{q_n} - \frac{jq_n + k}{q_nq_{n+1}} &< jq_n\theta + k\theta < jp_n + \frac{kp_n}{q_n} \\
jp_n + \frac{kp_n}{q_n} - \frac{1}{q_n} + \frac{1}{q_nq_{n+1}} &< jq_n\theta + k\theta < jp_n + \frac{kp_n}{q_n} \\
jp_n + \left\lfloor \frac{kp_n}{q_n} \right\rfloor + \left\{ \frac{kp_n}{q_n} \right\} - \frac{1}{q_n} + \frac{1}{q_nq_{n+1}} &< jq_n\theta + k\theta < jp_n + \left\lfloor \frac{kp_n}{q_n} \right\rfloor + \left\{ \frac{kp_n}{q_n} \right\}.
\end{aligned}$$

Because
$$0 < \left\{ \frac{kp_n}{q_n} \right\} - \frac{1}{q_n} + \frac{1}{q_n q_{n+1}} < 1$$
since
$$\min_{1 \leq k < q_n} \left\{ \frac{kp_n}{q_n} \right\} = \frac{1}{q_n}$$
we can write
$$0 < \frac{1}{q_n q_{n+1}} \leq \left\{ \frac{kp_n}{q_n} \right\} - \frac{1}{q_n} + \frac{1}{q_n q_{n+1}} < 1.$$
Thus, we have
$$jp_n + \left\lfloor \frac{kp_n}{q_n} \right\rfloor \leq jq_n\theta + k\theta < jp_n + \left\lfloor \frac{kp_n}{q_n} \right\rfloor + 1.$$
Another application of Lemma 2 gives,
$$\lfloor (jq_n + k)\theta \rfloor = jp_n + \left\lfloor \frac{kp_n}{q_n} \right\rfloor = jp_n + \lfloor k\theta \rfloor,$$
as desired. ∎

Lemma 8 will allow us to evaluate expressions involving the floor function. It should be noted that this problem has already been solved in full generality using the Ostrowski numeration system, and appears in (Brown and Shiue 1995, 184).

Properties of G

Let us consider the sequence $\{G(x, p_n, p_n, q_n)\}_{x=0}^{\infty}$, where the general term of the sequence is given by the formula

$$G(x, p_n, p_n, q_n) = \frac{2p_n x + p_n - q_n + p_n q_n - (-1)^n}{2} - \sum_{i=x+1}^{x+q_n} \lfloor i\theta \rfloor .$$

The above expression results when Corollary 1 is applied. The range of this sequence for $0 < \theta < 1$ is, up to sign, the set of natural numbers. Under the assumption that p_n/q_n is a convergent to θ, the number of occurrences of a given value in this sequence will be one of only two possible values. Let us explore this function in more detail.

Proposition 2 *Let p_n/q_n be a convergent of θ. Then $\{G(x, p_n, p_n, q_n)\}_{x=0}^{\infty}$ is a monotonically decreasing sequence if n is even, and a monotonically increasing sequence if n is odd.*

Proof. Recall that the floor function satisfies

$$\lfloor r \rfloor + \lfloor s \rfloor \le \lfloor r + s \rfloor \le \lfloor r \rfloor + \lfloor s \rfloor + 1$$

for any two positive real numbers r and s. Take n to be even and let $k \in \mathbb{N}_0$. Then $G(0, p_n, p_n, q_n) = -1$. Therefore, we have

$$\begin{aligned} G(k+1, p_n, p_n, q_n) - G(k, p_n, p_n, q_n) &= p_n - \lfloor (k+1+q_n)\theta \rfloor + \lfloor (k+1)\theta \rfloor \\ &= \lfloor q_n\theta \rfloor - \lfloor (k+1)\theta + q_n\theta \rfloor + \lfloor (k+1)\theta \rfloor . \end{aligned}$$

Because

$$\lfloor q_n\theta \rfloor + \lfloor (k+1)\theta \rfloor \le \lfloor q_n\theta + (k+1)\theta \rfloor ,$$

it follows that

$$\lfloor q_n\theta \rfloor - \lfloor (k+1)\theta + q_n\theta \rfloor + \lfloor (k+1)\theta \rfloor \le 0.$$

Hence,

$$G(k+1, p_n, p_n, q_n) - G(k, p_n, p_n, q_n) \le 0.$$

This demonstrates that our sequence is monotonically decreasing. Let n be odd and again let $k \in \mathbb{N}_0$. Then $G(0, p_n, p_n, q_n) = 1$. Thus, we have

$$\begin{aligned} G(k+1, p_n, p_n, q_n) - G(k, p_n, p_n, q_n) &= p_n - \lfloor (k+1+q_n)\theta \rfloor + \lfloor (k+1)\theta \rfloor \\ &= \lfloor q_n\theta \rfloor + 1 - \lfloor (k+1)\theta + q_n\theta \rfloor + \lfloor (k+1)\theta \rfloor . \end{aligned}$$

Because

$$\lfloor q_n\theta \rfloor + 1 + \lfloor (k+1)\theta \rfloor \geq \lfloor q_n\theta + (k+1)\theta \rfloor ,$$

it follows that

$$G(k+1, p_n, p_n, q_n) - G(k, p_n, p_n, q_n) \geq 0.$$

Hence,

$$G(k+1, p_n, p_n, q_n) - G(k, p_n, p_n, q_n) \geq 0.$$

This confirms that our sequence is monotonically increasing for the case n is odd. ■

If we take $p_n = q_n = 1$ with $\theta = \ln(2)/\ln(3)$, then

$$\{G(x, 1, 1, 1)\}_{x=0}^{\infty} = \{1, 1, 2, 2, 2, 3, 3, 3, 4, 4, 5, 5, 5, 6, 6, 6, 7, 7, \ldots\} .$$

The set of all values of the sequence $\{G(x, p_n, p_n, q_n)\}_{x=0}^{\infty}$ forms a multiset, $M = \{r_1 \cdot x_1, r_2 \cdot x_2, \ldots\}$ where r_i is the repetition number for the element x_i. The elements of M will all be positive if n is odd and all negative if n is even.

Definition 1 *Let us now define a function*

$$C_{\frac{p_n}{q_n}}(x_i) = r_i$$

which will count the number of repetitions of the value x_i *in the sequence* $\{G(x, p_n, p_n, q_n)\}_{x=0}^{\infty}$.

Using the above example, we have

$$\left\{C_{\frac{1}{1}}(x)\right\}_{x=1}^{\infty} = \{2, 3, 3, 2, 3, 3, 2, ...\}.$$

Relation to Sturmian Sequences

Since

$$\begin{aligned} G(x+1,p_n,p_n,q_n) - G(x,p_n,p_n,q_n) &= p_n + \sum_{j=1}^{x} f_\theta(j) - \sum_{i=1}^{x+q_n} f_\theta(i) \\ &= p_n - \sum_{i=x+1}^{x+q_n} f_\theta(i), \end{aligned}$$

it is clear that $G(x+1,p_n,p_n,q_n) - G(x,p_n,p_n,q_n) = \pm 1$ when the number of 1's in the subword u whose first element position is $x+1$ of length q_n satisfies $|u|_1 = p_n \pm 1$. It can, and will, be shown that exactly p_n 1's occur in the first characteristic block of f_θ so that $G(1,p_n,p_n,q_n) - G(0,p_n,p_n,q_n) = 0$.

Example 1 *If we consider the first few elements of the characteristic sequence*

$$f_\theta = 101101101011011\cdots,$$

then $G(1,2,2,3) - G(0,2,2,3) = 0$ because the sum of the first three elements of f_θ.is 2 and $p_3 = 2$.

Lemma 9 *Let X_n be the prefix of length q_n of f_θ. Then $|X_n|_1 = p_n$.*

Proof. Since the prefix X_n consists of only 0's and 1's, it follows that

$$|X_n|_1 = \sum_{i=1}^{q_n} f_\theta(i).$$

However, it is also true that

$$\sum_{i=1}^{q_n} f_\theta(i) = \lfloor (q_n+1)\theta \rfloor .$$

Thus,

$$\lfloor (q_n+1)\theta \rfloor = p_n$$

is what needs to be proven. When n is even,

$$\begin{aligned}
0 &< q_n\theta - p_n < \frac{1}{q_{n+1}} \\
p_n &< q_n\theta < p_n + \frac{1}{q_{n+1}} \\
p_n + \theta &< q_n\theta + \theta < p_n + \theta + \frac{1}{q_{n+1}}.
\end{aligned}$$

Since

$$p_{n+1} + 1 \leq q_{n+1},$$

we have

$$\begin{aligned}
q_{n+1}\theta &< p_{n+1} \\
1 + q_{n+1}\theta &< p_{n+1} + 1 \leq q_{n+1} \\
1 + q_{n+1}\theta &\leq q_{n+1} \\
\frac{1}{q_{n+1}} + \theta &\leq 1 \\
p_n + \theta &< q_n\theta + \theta < p_n + 1 \\
p_n &\leq q_n\theta + \theta < p_n + 1.
\end{aligned}$$

Therefore, we conclude that

$$\lfloor q_n\theta + \theta \rfloor = p_n.$$

For the case when n is odd and $n \geq 1$,

$$\begin{aligned}
\frac{-1}{q_{n+1}} &< q_n\theta - p_n < 0 \\
p_n - \frac{1}{q_{n+1}} &< q_n\theta < p_n \\
p_n + \theta - \frac{1}{q_{n+1}} &< q_n\theta + \theta < p_n + \theta.
\end{aligned}$$

Since

$$\begin{aligned}
1 &< p_{n+1} < q_{n+1}\theta \\
\frac{1}{q_{n+1}} &< \theta \\
0 &< \theta - \frac{1}{q_{n+1}} < \theta,
\end{aligned}$$

we have

$$\begin{aligned} p_n &< q_n\theta + \theta < p_n + \theta \\ p_n &\leq q_n\theta + \theta < p_n + 1. \end{aligned}$$

Therefore, we conclude that

$$\lfloor q_n\theta + \theta \rfloor = p_n.$$

This completes the proof. As an immediate consequence of Lemma 9, we find that

$$|X_n|_0 = q_n - p_n.$$

Since the prefix X_n has q_n elements and satisfies $|X_n|_1 = p_n$, it follows that $q_n - |X_n|_1 = p_n$. Thus far, we have established that

$$G(1, p_n, p_n, q_n) - G(0, p_n, p_n, q_n) = 0,$$

and that the sequence

$$\{G(x, p_n, p_n, q_n)\}_{x=0}^{\infty}$$

is monotone. Combining these facts with the additional fact that the characteristic sequence is a Sturmian sequence (which is balanced), the value of

$$G(x+1, p_n, p_n, q_n) - G(x, p_n, p_n, q_n) = \begin{cases} 0 \text{ or } 1, & \text{for } n \text{ odd} \\ 0 \text{ or } -1, & \text{for } n \text{ even} \end{cases}.$$

We will now continue our investigation of $G(x)$. ■

First Value of $C_{p_n/q_n}(x)$

For now though, let's find the general formula for $G(x+1, p_n, p_n, q_n) - G(x, p_n, p_n, q_n)$. For $x \geq 0$,

$$\begin{aligned} G(x+1, p_n, p_n, q_n) - G(x, p_n, p_n, q_n) &= p_n - \sum_{i=x+2}^{x+1+q_n} \lfloor i\theta \rfloor + \sum_{i=x+1}^{x+q_n} \lfloor i\theta \rfloor \\ &= p_n + \lfloor (x+1)\theta \rfloor - \lfloor (x+1+q_n)\theta \rfloor. \end{aligned}$$

It is not hard to find particular values of x that satisfy $G(x+1) - G(x) = \pm 1$.

Lemma 10 *Let $x = q_{n+1} - 1$. Then*

$$G(x+1, p_n, p_n, q_n) - G(x, p_n, p_n, q_n) = \begin{cases} 1 & , \text{ for } n \text{ odd} \\ -1 & , \text{ for } n \text{ even} \end{cases} .$$

Proof. Let n be odd. We require,

$$\begin{aligned} G(q_{n+1}, p_n, p_n, q_n) - G(q_{n+1} - 1, p_n, p_n, q_n) &= 1 \\ p_n - \sum_{i=q_{n+1}+1}^{q_{n+1}+q_n} \lfloor i\theta \rfloor + \sum_{j=q_{n+1}}^{q_{n+1}+q_n-1} \lfloor j\theta \rfloor &= 1 \\ p_n - \lfloor (q_{n+1} + q_n)\theta \rfloor + \lfloor q_{n+1}\theta \rfloor &= 1 \\ \lfloor (q_{n+1} + q_n)\theta \rfloor &= p_{n+1} + p_n - 1. \end{aligned}$$

An application of Lemma 7 to the following expression

$$\lfloor (q_{n+1} + q_n)\theta \rfloor$$

confirms the result. Let n be even. We require,

$$\begin{aligned} G(q_{n+1}, p_n, p_n, q_n) - G(q_{n+1} - 1, p_n, p_n, q_n) &= -1 \\ p_n - \sum_{i=q_{n+1}+1}^{q_{n+1}+q_n} \lfloor i\theta \rfloor + \sum_{j=q_{n+1}}^{q_{n+1}+q_n-1} \lfloor j\theta \rfloor &= -1 \\ p_n - \lfloor (q_{n+1} + q_n)\theta \rfloor + \lfloor q_{n+1}\theta \rfloor &= -1 \\ \lfloor (q_{n+1} + q_n)\theta \rfloor &= p_{n+1} + p_n. \end{aligned}$$

Another application of Lemma 7 to the expression

$$\lfloor (q_{n+1} + q_n)\theta \rfloor$$

completes the proof. ■

So far, we have shown that

$$G(0, p_n, p_n, q_n) = (-1)^{n+1}$$

and that

$$G(q_{n+1}, p_n, p_n, q_n) - G(q_{n+1} - 1, p_n, p_n, q_n) = (-1)^{n+1}.$$

At this point, however, we cannot be sure that there are values of x between the values just determined. What must be demonstrated is that $G(x)$ is constant on certain intervals whose endpoints are the values that have just been determined.

Properties of Prefixes

We now need another lemma that tells us about the structure of the reversal of two prefixes.

Lemma 11 *Consider the function* c *which switches the last two symbols of a word and leaves the rest of the word unchanged. Then*

$$X_n X_{n-1} = c(X_{n-1} X_n).$$

Proof. This proof is due to (Allouche and Shallit 2003, 288). For $n = 1$, observe that

$$\begin{aligned} X_1 X_0 &= 0^{a_1-1}10 \\ &= c(0^{a_1-1}01) \\ &= c(00^{a_1-1}1) \\ &= c(X_0 X_1). \end{aligned}$$

Now assume the result is true for all k satisfying $1 \leq k < n$, where $n \geq 2$. Then

$$\begin{aligned} X_n X_{n-1} &= (X_{n-1}^{a_n} X_{n-2}) X_{n-1} \\ &= X_{n-1}^{a_n} c(X_{n-1} X_{n-2}) \\ &= c(X_{n-1}^{a_n} X_{n-1} X_{n-2}) \\ &= c(X_{n-1} X_{n-1}^{a_n} X_{n-2}) \\ &= c(X_{n-1} X_n). \end{aligned}$$

This completes the proof of Lemma 11. ■

The fascinating aspect of Lemma 11 shows that every prefix $X_n X_{n-1}$ enjoys a "quasi" commutative property. The "quasi" arises from the fact that the last two symbols must be different.

Lemma 12 *The strings* $X_{n+1} X_n$ *and* $X_{n+2} X_{n+1} X_n$ *are prefixes of* f_θ*.*

Proof. Recall, by definition, that the prefix X_n is a prefix of X_{n+1}. Consider the prefix $X_{n+2} = X_{n+1}^{a_{n+2}} X_n$. If $a_{n+2} = 1$, then we are done. If $a_{n+2} > 1$, then minimally

$$\begin{aligned} X_{n+2} &= X_{n+1} X_{n+1} X_n \\ &= X_{n+1} X_n^{a_{n+1}} X_{n-1} X_n \end{aligned}$$

which proves the first part of the lemma. The second part can be demonstrated by considering the prefix $X_{n+4} = X_{n+3}^{a_{n+4}} X_{n+2}$. Suppose first that $a_{n+4} > 1$. Then

$$\begin{aligned} X_{n+4} &= X_{n+3} X_{n+3}^{a_{n+4}-1} X_{n+2} \\ &= X_{n+2}^{a_{n+3}} X_{n+1} X_{n+3}^{a_{n+4}-1} X_{n+2}. \end{aligned}$$

If $a_{n+3} = 1$, then we are done. If $a_{n+3} > 1$, then

$$\begin{aligned} X_{n+4} &= X_{n+2} X_{n+2}^{a_{n+3}-1} X_{n+1} X_{n+3}^{a_{n+4}-1} X_{n+2} \\ &= X_{n+2} (X_{n+1}^{a_{n+2}} X_n)^{a_{n+3}-1} X_{n+1} X_{n+3}^{a_{n+4}-1} X_{n+2}, \end{aligned}$$

which yields either

$$X_{n+4} = X_{n+2} (X_{n+1} X_n)^{a_{n+3}-1} X_{n+1} X_{n+3}^{a_{n+4}-1} X_{n+2},$$

or

$$X_{n+4} = X_{n+2} \left[(X_{n+1} X_{n+1}^{a_{n+2}-1}) X_n \right]^{a_{n+3}-1} X_{n+1} X_{n+3}^{a_{n+4}-1} X_{n+2},$$

depending on whether $a_{n+2} = 1$ or $a_{n+2} > 1$, respectively. In either case, $X_{n+2} X_{n+1} X_n$ is a prefix of X_{n+4}, which is a prefix of f_θ. Having considered all possible cases when $a_{n+4} > 1$, let us focus on the case when $a_{n+4} = 1$. In this case,

$$\begin{aligned} X_{n+4} &= X_{n+3} X_{n+2} \\ &= X_{n+2}^{a_{n+3}} X_{n+1} X_{n+2}. \end{aligned}$$

If $a_{n+3} = 1$, then we are done. If $a_{n+3} > 1$, then

$$X_{n+4} = X_{n+2} (X_{n+1}^{a_{n+2}} X_n)^{a_{n+3}-1} X_{n+1} X_{n+2},$$

which yields either

$$X_{n+4} = X_{n+2} (X_{n+1} X_n)^{a_{n+3}-1} X_{n+1} X_{n+2}$$

or

$$X_{n+4} = X_{n+2} (X_{n+1} X_{n+1}^{a_{n+2}-1} X_n)^{a_{n+3}-1} X_{n+1} X_{n+2},$$

depending on whether $a_{n+2} = 1$ or $a_{n+2} > 1$, respectively. In either case, $X_{n+2} X_{n+1} X_n$ is a prefix of X_{n+4}, which is a prefix of f_θ. This completes the proof. ■

Lemma 13 *Every subword u having length q_n of X_{n+1} satisfies $|u|_1 = p_n$. The same holds for $X_n^{a_j}$ where $j \geq 1$.*

Proof. If we consider the following expression for $X_n^{a_j}$,

$$X_n^{a_j} = \underbrace{[f_\theta(1)f_\theta(2)f_\theta(3)\cdots f_\theta(q_n)]\cdots[f_\theta(1)f_\theta(2)f_\theta(3)\cdots f_\theta(q_n)]}_{a_j \text{ times}},$$

then it can be seen that every subword u must begin with the last m elements of X_n for some m with $0 \leq m \leq q_n$. Since $|u| = q_n$, it follows that the last $q_n - m$ elements of u must be precisely the first $q_n - m$ elements of X_n. Apart from order, the subword u comprises the last m elements of X_n and the first $q_n - m$ elements of X_n. Thus, the subword u contains precisely the same elements as X_n. It follows that the subword u satisfies

$$|u|_1 = |X_n|_1 .$$

Applying Lemma 9 yields $|u|_1 = p_n$. Now consider taking $a_j = a_{n+1}$ so that

$$X_{n+1} = X_n^{a_{n+1}} X_{n-1}.$$

It is sufficient now to consider only

$$X_n X_{n-1} = [f_\theta(1)f_\theta(2)f_\theta(3)\cdots f_\theta(q_n)]\,[f_\theta(1)f_\theta(2)f_\theta(3)\cdots f_\theta(q_{n-1})]$$

since every subword u contained in $X_n^{a_{n+1}-1}$ satisfies $|u|_1 = p_n$. Let u begin with the last m elements of X_n, where $q_n - q_{n-1} \leq m \leq q_n$. The last $q_n - m$ elements must be the first $q_n - m$ elements of X_n since X_{n-1} is a prefix of X_n. Apart from order, the subword u comprises the last m elements of X_n and the first $q_n - m$ elements of X_n. It follows that the subword u satisfies $|u|_1 = p_n$. This completes the proof. ■

Remark 2 *The last lemma could also be proved by utilizing the fact that any set of k consecutive integers is equal to the set of the first k positive integers when reduced modulo k.*

Lemma 14 *The subsequence*

$$\{G(x, p_n, p_n, q_n)\}_{x=0}^{x=q_{n+1}-1}$$

is constant.

Proof. Consider the prefix

$$X_{n+1}X_n = X_n^{a_{n+1}}X_{n-1}X_n.$$

If we now interchange the last two elements of X_{n-1}, then we can write

$$X_{n+1}X_n = X_n^{a_{n+1}}X_nX_{n-1}^*.$$

Given the structure of this prefix, we can see that every subword u whose first element occurs at position j, where $j \leq q_{n+1} - 1$ must satisfy $|u|_1 = p_n$, since the last two elements of the string X_{n-1}^* are not included in the set of subwords under consideration. ■

Upon examination, it appears that the first element of $C_{p_n/q_n}(x)$ occurs exactly a_{n+2} times, as illustrated by the following table.

Table 2: First Repetition in C

p_n/q_n	1/1	1/2	2/3	5/8	12/19	41/65
a_{n+2}	1	2	2	3	1	5

Let's focus on proving that this is indeed the case.

Theorem 2 *The first value that appears in C_{p_n/q_n} is q_{n+1}. Moreover, this value repeats a_{n+2} times.*

Proof. It was proved in the last lemma that $\{G(x, p_n, p_n, q_n)\}_{x=0}^{x=q_{n+1}-1}$ is constant. Also, it was shown in Lemma 10 that

$$G(q_{n+1}, p_n, p_n, q_n) - G(q_{n+1} - 1, p_n, p_n, q_n) = \pm 1.$$

Therefore, $\{G(x, p_n, p_n, q_n)\}_{x=0}^{x=q_{n+1}-1}$ contains q_{n+1} identical elements. Therefore, we have $C_{p_n/q_n}(x_1) = q_{n+1}$. To demonstrate the second part, consider the prefix

$$X_{n+2} = X_{n+1}^{a_{n+2}}X_n = \underbrace{(X_{n+1})\cdots(X_{n+1})}_{a_{n+2}\text{ times}}X_n.$$

We know from our previous work that every subword u of X_{n+1} satisfies the equation $|u|_1 = p_n$ when $|u| = q_n$. Observe that when the prefixes X_{n+1} are juxtaposed, the subword u with first element position at $x = q_{n+1}$ must satisfy $|u|_1 \neq p_n$. To see that this is indeed the case, consider the factorization

$$X_{n+1}X_{n+1} = X_n^{a_{n+1}} X_{n-1} X_n^{a_{n+1}} X_{n-1}.$$

If we utilize the upcoming Theorem 3, then based on the above factorization, the subword u will force a variation in G when $x = q_{n+1} - 1$. It is clear now that

$$X_{n+2} = X_{n+1}^{a_{n+2}} X_n = \underbrace{(X_{n+1}) \cdots (X_{n+1})}_{a_{n+2} \text{ times}} X_n$$

contains a_{n+2} subwords that cause a variation in G. To finish the proof, we need to show that q_{n+1} does not occur $a_{n+2} + 1$ times. Consider the extended prefix

$$\begin{aligned} X_{n+2} X_{n+1} X_n &= X_{n+1}^{a_{n+2}} X_n X_{n+1} X_n \\ &= X_{n+1}^{a_{n+2}} (X_n X_n^{a_{n+1}} X_{n-1}) X_n \end{aligned}$$

which means $x = q_n + q_{n+1}$ must be the second distinct value that occurs in $\left\{C_{p_n/q_n}(x)\right\}_{x=x_1}^{\infty}$. ■

The Second Distinct Value of $C_{p_n/q_n}(x)$

In Section 3 we saw that for every convergent, $C_{p_n/q_n}(x_1) = q_{n+1}$. Beyond the first element, these sequences appear to behave differently. Let us consider a few examples.

It appears that each of these sequences has a structure similar to that of a Sturmian sequence, although this must be left for future research.

Remark 3 *Consider the difference*

$$G(x+1, p_n, p_n, q_n) - G(x, p_n, p_n, q_n) = p_n + \lfloor (x+1)\theta \rfloor - \lfloor (x+1+q_n)\theta \rfloor ,$$

Table 3: Values of C

p_n/q_n	$\{C_{p_n/q_n}(x)\}_{x=x_1}^{\infty}$
1/1	$\{2, 3, 3, 2, 3, 3, 2, 3, 3, 3, \ldots\}$
1/2	$\{3, 3, 5, 3, 5, 3, 3, 5, 3, 5, \ldots\}$
2/3	$\{8, 8, 11, 8, 11, 8, 11, 8, 8, 11, \ldots\}$
5/8	$\{19, 19, 19, 27, 19, 19, 19, 27, 19, 19, \ldots\}$

where n is odd. Now let $x+1 = q_{2k} > n$. Then,

$$\begin{aligned} p_n + \lfloor q_{2k}\theta \rfloor - \lfloor (q_{2k}+q_n)\theta \rfloor &= p_n + p_{2k} - (p_{2k} + p_n - 1) \\ &= 1. \end{aligned}$$

If $x+1 = q_{2k+1} > n$, then we have

$$\begin{aligned} p_n + \lfloor q_{2k+1}\theta \rfloor - \lfloor (q_{2k+1}+q_n)\theta \rfloor &= p_n + p_{2k+1} - 1 - (p_{2k+1} + p_n - 1) \\ &= 0. \end{aligned}$$

Since this is neither a strong nor an interesting result, we shall have to use other means to determine the values of x for which $G(x+1) - G(x) = 0$.

Analysis Using Combinatorics on Words

Theorem 3 *In the string $X_{n-1}X_n$, if the first element of the subword u is $q_{n-1} - 1$ at position x, then*

$$G(x+1, p_n, p_n, q_n) - G(x, p_n, p_n, q_n) = (-1)^{n+1}.$$

Proof. Consider the subword $X_{n-1}X_n$. This subword is the same as X_nX_{n-1} except for the last two symbols, which have been interchanged. Notice that the last word of length q_n must satisfy $|X_n|_1 = p_n$. Moreover, all other subwords of length q_n agree exactly with X_nX_{n-1}, except possibly the next to last subword.

By our previous work, we know that the last two symbols of X_n are not equal. For, when n is odd,

$$f_\theta(q_n) = \lfloor (q_n + 1)\theta \rfloor - \lfloor q_n\theta \rfloor = p_n - (p_n - 1) = 1$$

and

$$f_\theta(q_n - 1) = \lfloor q_n\theta \rfloor - \lfloor (q_n - 1)\theta \rfloor = p_n - 1 - (p_n - 1) = 0.$$

Also, when n is even,

$$f_\theta(q_n) = \lfloor (q_n + 1)\theta \rfloor - \lfloor q_n\theta \rfloor = p_n - p_n = 0$$

and

$$f_\theta(q_n - 1) = \lfloor q_n\theta \rfloor - \lfloor (q_n - 1)\theta \rfloor = p_n - (p_n - 1) = 1.$$

Therefore, only the next to the last subword of length q_n of $X_{n-1}X_n$ can, and must, have a different sum. Given the fact that G is monotone, this completes the proof. ■

Theorem 4 *The second distinct value of $C_{p_n/q_n}(x)$ is $q_{n+1} + q_n$.*

Proof. First, we need to find a prefix which is long enough and has a form that facilitates the analysis. The prefix we seek is:

$$X_{n+2}X_{n+1}X_n = X_{n+1}^{a_{n+2}} X_n X_{n+1} X_n.$$

Notice now that the first term is

$$X_{n+1}^{a_{n+2}} = \underbrace{X_{n+1} \cdots X_{n+1}}_{a_{n+2} \text{ times}},$$

and is followed by X_n. Through our previous work, we know that $G(x)$ is constant for all x with $0 \leq x \leq q_{n+1} - 1$. Given the fact that X_{n+1} is repeated a_{n+2} times shows that the first a_{n+2} elements of $C_{p_n/q_n}(x)$ are equal. It suffices now to consider the truncated prefix $X_n X_{n+1} X_n$. It is an easy observation that every subword u of $X_n X_{n+1}$ of length q_n satisfies $|u|_1 = p_n$. This is true since it can be rewritten as $X_n X_n^{a_{n+1}} X_{n-1}$,.so that Lemma 13 applies. It has already been demonstrated that a variation in the sequence G occurs at $x = q_{n+1} - 1$ with respect to the prefix $X_{n+1}X_n$. Therefore, the second value that occurs in $C_{p_n/q_n}(x)$ must be $q_{n+1} + q_n$. ■

Relation to the Three Distance Theorem

There seems to be a connection between the results obtained in the previous sections of this thesis and the three distance theorem. In fact, it seems very likely that a proof of the three distance theorem could be constructed using only the properties of Sturmian words. Several different proofs of the three distance theorem using combinatorics on words appear in (Alessandri and Berthe 1998). In the next section, the three distance theorem and a numerical example will be given.

The Three Distance Theorem

Suppose we introduce a sequence $\{s_n\}_{n=1}^{\infty}$, where $s_n = \{n\theta\}$. This sequence is commonly referred to as a fractional part sequence. Notice that s_n may be rewritten as $s_n = n\theta - \lfloor n\theta \rfloor$. If θ is a positive irrational real number, then the k points generated by $\{s_n\}_{n=1}^{k}$ will partition the unit circle into $k+1$ intervals. The three distance theorem states that a minimum of two and a maximum of three distinct interval lengths will occur. If three lengths do occur, then one will be the sum of the other two.

Example 2 *The following figure shows a polar plot of the first forty terms of the sequence* $\{s_n\}_{n=1}^{\infty}$ *using* $\theta = \ln(2)/\ln(3)$.

The set of gaps for this particular value of n is:

$$\{0.012335\cdots, 0.02277\cdots, 0.03510\cdots\}.$$

Computing $\{G(x,2,2,4)\}_{x=0}^{x=20}$ reveals that the associated sequence

$$\left\{C_{2/4}(x)\right\}_{x=-2}^{x=-7} = \{1,2,1,2,3,2\}.$$

Similar to the previous example, three gaps occur and one is the sum of the other two. It is interesting to notice the similarity between the value

$$G(x,1,1,1) = x - \lfloor (x+1)\theta \rfloor + \left\lfloor \frac{1}{\theta} \right\rfloor$$

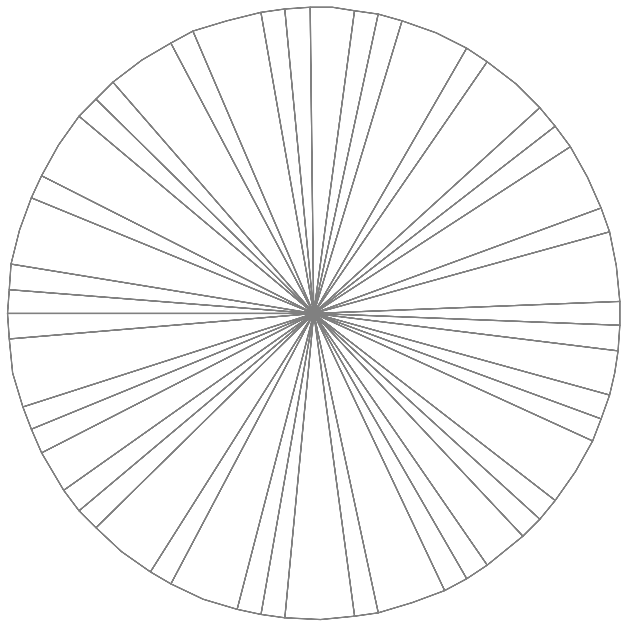

Figure 1: Three distance theorem illustration.

and the nth term of the sequence

$$s_n = n\theta - \lfloor n\theta \rfloor .$$

While it is clear that $G(x, 1, 1, 1)$ and s_n are not of the same form, they do seem to exhibit similar behavior in terms of the quantity of distinct values that occur in them.

The Structure of $C_{p_n/q_n}(x)$

It appears that there is a simple way to construct the sequences $\left\{C_{p_n/q_n}(x)\right\}_{x=x_1}^{\infty}$ without actually counting the elements repeated in the sequence $\{G(x, p_n, p_n, q_n)\}_{x=x_1}^{\infty}$.

If we take our standard example of

$$\theta = \frac{\ln(2)}{\ln(3)} = [0, 1, 1, 1, 2, 2, 3, 1, \ldots],$$

then let us resolve the prefix X_5 into a string of prefixes using only X_1 and X_2. Indeed,

$$\begin{aligned} X_5 &= X_4X_4X_3 \\ &= (X_3X_3X_2)(X_3X_3X_2)X_3 \\ &= (X_2X_1)(X_2X_1)X_2(X_2X_1)(X_2X_1)X_2(X_2X_1). \end{aligned}$$

Consider now the substitution scheme

$$X_2X_1 = q_1 + q_2 = 3$$

and

$$X_2X_2X_1 = q_1 + 2 \cdot q_2 = 5.$$

Applying this substitution yields,

$$\begin{aligned} X_5 &= (X_2X_1)(X_2X_1)(X_2X_2X_1)(X_2X_1)(X_2X_2X_1) \\ &= 33535, \end{aligned}$$

which agrees with $\left\{C_{1/2}(x)\right\}_{x=1}^{x=5}$ This leads us to the following conjecture.

Conjecture 1 *The sequence $\left\{C_{p_n/q_n}(x)\right\}_{x=x_1}^{\infty}$ is precisely the sequence formed by resolving the sequence f_θ into the prefixes X_{n-1} and X_n and applying the substitution*

$$X_n^{a_{n+1}}X_{n-1} = q_{n+1}$$

and

$$X_n^{a_{n+1}+1}X_{n-1} = q_n + q_{n+1}.$$

Remark 4 *It is interesting to notice that given the form of the substitution stated above, it is not necessary to resolve X_k into the prefixes X_{n-1} and X_n, but only the prefixes X_n and X_{n+1}.*

Constant values of $G(x, y, mp_n, mq_n)$

From examination of the interspersion that was introduced in Section 1 regarding the distribution of numbers of the form $a^x b^y$, it is clear that there must exist positions in the rank array which satisfy

$$R(x, y) - R(x + mq, y - mp) = m\,[R(x, y) - R(x + q, y - p)]$$

for a given positive integer m. An interesting question to ask then is: How long can such a sequence be and at what positions does it begin and end?

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APPENDICES

The Sequence $f_{\ln(2)/\ln(3)}$

The first thirty terms of the characteristic sequence f_θ for $\theta = \ln(2)/\ln(3)$ are

$$f_\theta = 101101101011011010110110110101 \cdots .$$

Sequences $\{G(x,p_n,p_n,q_n)\}_{x=0}^{x=19}$

Listed below are the first twenty terms of the sequences $\{G(x,p_n,p_n,q_n)\}_{x=0}^{x=19}$ where $1 \le n \le 4$.

$$\begin{aligned}
\{G(x,1,1,1)\}_{x=0}^{x=19} &= \{1,1,2,2,2,3,3,3,4,4,5,5,5,6,6,6,7,7,8,8\} \\
\{G(x,1,1,2)\}_{x=0}^{x=19} &= \{-1,-1,-1,-2,-2,-2,-3,-3,-3,-3, \\
&\quad -3,-4,-4,-4,-5,-5,-5,-5,-5,-6\} \\
\{G(x,2,2,3)\}_{x=0}^{x=19} &= \{1,1,1,1,1,1,1,1,1,2,2,2,2,2,2,2,2,3,3,3,3\} \\
\{G(x,5,5,8)\}_{x=0}^{x=19} &= \{-1,-1,-1,-1,-1,-1,-1,-1,-1,-1, \\
&\quad -1,-1,-1,-1,-1,-1,-1,-1,-1,-2\}
\end{aligned}$$

Sequences $\left\{C_{p_n/q_n}(x)\right\}_{x=1}^{x=20}$

Listed below are the first twenty values of the sequences $\left\{C_{p_n/q_n}(x)\right\}_{x=1}^{x=20}$ where $1 \leq n \leq 4$.

$$\begin{aligned}
\left\{C_{\frac{1}{1}}(x)\right\}_{x=1}^{x=20} &= \{2,3,3,2,3,3,2,3,3,3,2,3,3,2,3,3,3,2,3,3\} \\
\left\{C_{\frac{1}{2}}(x)\right\}_{x=1}^{x=20} &= \{3,3,5,3,5,3,3,5,3,5,3,3,5,3,5,3,3,5,3,5\} \\
\left\{C_{\frac{2}{3}}(x)\right\}_{x=1}^{x=20} &= \{8,8,11,8,11,8,11,8,8,11,8,11,8,11,8,11,8,8,11,8\} \\
\left\{C_{\frac{5}{8}}(x)\right\}_{x=1}^{x=20} &= \{19,19,19,27,19,19,19,27,19,19,19,27,19,19,19,27,19,19,19,27\}
\end{aligned}$$