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The Kumaraswamy Inverse Weibull Poisson Distribution With Applications

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THE KUMARASWAMY INVERSE WEIBULL POISSON DISTRIBUTION WITH APPLICATIONS

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The aim of this thesis is to propose a new distribution called the Kumaraswamy inverse Weibull Poisson (KIWP) Distribution. The distribution properties including hazard functions, reverse hazard functions, survival functions, quantile functions, moments, distributions of order statistics, mean deviations, Lorenz and Bonferroni curves and Fischer information are presented. The maximum likelihood method is used to estimate the model parameters of this new distribution. The special cases of the KIWP distribution including the inverse Weibull Poisson (IWP), Kumaraswamy Frechet Poisson (KFP), Kumaraswamy inverse exponentiated Poisson (KIEP) and Kumaraswamy inverse Rayleigh Poisson (KIRP) distributions are presented. A Monte Carlo simulation study is presented to exhibit the performance and accuracy of the maximum likelihood estimates of the KIWP model parameters. Real data examples are used to show the usefulness of the proposed model.

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INTRODUCTION

Introduction

Statistical lifetime distributions are employed broadly in data modeling. They are greatly utilized in areas such as reliability engineering, survival analysis, social sciences and a host of other applications. Of particular interest are applications of lifetime distributions in reliability engineering, including the measurement of survival time of electrical components. In medicine, an interesting application of lifetime distributions is in the calculation of survival times of patients post surgery. Khan et al. (2008) defined reliability as the probability that a system or process performs its prescribed duty without failure for a given time given it is operated correctly in a specified environment. In social sciences, an interesting application is modelling the lifetime of marriages, Almalki and Nadarajah (2014). One of the most important distributions used in modeling lifetime data is the Weibull distribution whose cumulative distribution function (cdf) and probability density function (pdf) are respectively given by

$$F(x) = 1 - \exp(-\alpha x^{\beta})$$

and

$$f(x) = \alpha\beta x^{\beta-1}\exp(-\alpha x^{\beta}),$$

where $x > 0, \alpha > 0$, and $\beta > 0$. Notice that α is a scale parameter and β is a shape parameter.

Liu (1997) explained the use of the Weibull distribution as being an appropriate model of failure in instances where an item consists of numerous components and each component has an identical failure time distribution and the item fails when the weakest part fails. If X is a random variable following a Weibull Distribution with parameters $\alpha, \beta > 0$, under the transformation $1/X$, the inverse Weibull distribution proposed by Keller and Kamath (1982) is obtained. The cdf and pdf of the inverse Weibull distribution are respectively given by

$$F(x;\alpha,\beta) = \exp(-\alpha x^{-\beta}), \quad x > 0, \alpha > 0, \beta > 0, \tag{1.1}$$

$$f(x;\alpha,\beta) = \alpha\beta x^{-\beta-1}\exp(-\alpha x^{-\beta}). \quad x > 0, \alpha > 0, \beta > 0 \tag{1.2}$$

The hazard function of the inverse Weibull distribution is given by:

$$h(x,\alpha,\beta) = \frac{\beta\alpha^{-\beta}x^{-\beta-1}\exp[-(\alpha x)^{-\beta}]}{1-\exp[-(\alpha x)^{-\beta}]}$$

The inverse Weibull distribution has proved to be very useful in the modeling of lifetime data. It has been compounded with other continuous distributions to produce new lifetime distributions.

Pararai et al. (2014) proposed the gamma inverse Weibull distribution by compounding the inverse Weibull distribution density function with the gamma generator proposed by Ristić and Balakrishnan (2012). The gamma inverse Weibull distribution is a 3 parameter distribution with parameters λ, β and σ. Some of the submodels of the gamma inverse Weibull (GIW) include the inverse Weibull distribution, the gamma Frechet (GF) distribution and the inverse Rayleigh (IR) distribution. The hazard rate function assumes unimodal and upside down bathtub shapes. The cdf and pdf of the GIW distribution are respectively given by

$$F_{GIW}(x) = 1 - \frac{\gamma(-\log(\exp[-(\alpha x)^{-\beta}]),\delta)}{\Gamma(\delta)}$$

then,

$$g_{GIW}(x) = \frac{\beta x^{-1}}{\Gamma(\delta)}\left[\lambda x^{-\beta}\right]^{\delta}\exp\left[-\lambda x^{-\beta}\right].$$

where $\lambda > 0, \beta > 0$, and $\delta > 0$. The GIW distribution was applied to a data set from Bjerkedal et al. (1960) on the survival times in days of guinea pigs injected with different doses of tubercle bacilli. The GIW distribution was also applied to a data set from Lawless (1982) on the number of millions of revolutions before failure of each of 23 ball bearings in a life testing experiment. The GIW was found to be superior to all the models it was compared against.

The Kumaraswamy inverse Weibull (KIW) distribution was proposed and studied by Shahbaz et al. (2012). Its cdf and pdf are respectively given by

$$F_{KIW}(x) \;=\; 1 - \left[1 - \exp(-a\alpha x^{-\beta})\right]^{b}$$

and

$$\begin{aligned} f_{KIW}(x) &= ab\alpha\beta x^{-\beta-1}\exp(-\alpha x^{-\beta})\left[\exp(-\alpha x^{-\beta})\right]^{a-1} \\ &\times \left[1 - \{\exp(-\alpha x^{-\beta})\}^{a}\right]^{b-1}. \end{aligned}$$

The KIW distribution was applied to survival time data and salary data by Shahbaz et al. (2012). Cordeiro and de Castro (2011) proposed a generalized class of distribution called the Kum-G distribution which has a cdf of the following form

$$F_{K-G}(x) = 1 - [1 - G(x)^{a}]^{b}$$

and the corresponding pdf of the generalized class of distribution is as follows

$$f_{Kum-G}(x) = abg(x)G(x)^{a-1}[1 - G(x)^{a}]^{b-1}.$$

Shahbaz et al. (2012) substituted in the cumulative distribution function of the inverse Weibull distribution into the probability density function of the Kum-G distribution to obtain the Kumaraswamy inverse Weibull distribution. The beta inverse Weibull distribution was proposed and studied by Khan (2010). Hanook et al. (2013) further investigated the properties of the beta inverse Weibull distribution which is given by:

$$G_{(x)} = \frac{B_{F(x)}(a,b)}{B(a,b)}$$

where

$$B(a,b) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}$$

and F(x) is the baseline cdf. The cdf of the BIW distribution is given by

$$F_{BIW} = \int_0^{GIW} t^{a-1}(1-t)dt.$$

Another interesting modification of the inverse Weibull distribution is the length biased inverse Weibull distribution proposed by Kersey and Oluyede (2013). Consider the weight function $\omega(x) = x$ and the inverse Weibull distribution with the pdf given in 1.2. The general form of the length biased inverse Weibull (LBIW) pdf is given by:

$$g_\omega(x,\alpha,\beta) = \frac{f(x,\alpha,\beta)}{\mu_F}$$

where $x \geq 0$, $\alpha > 0$, $\beta > 1$ and μ_F is the mean. The corresponding cdf for the LBIW distribution is given by:

$$G_{LBIW} = \frac{\beta\alpha^{-\beta+1}}{\Gamma(1-1/\beta)} \int_0^x t^{-\beta} \exp(-(\alpha x)^{-\beta})dt.$$

The hazard function of the LBIW is given by:

$$h(x,\alpha,\beta) = \frac{x^{-\beta \exp(-(\alpha x)^{-\beta})}}{\int_x^\infty t^{-\beta} \exp(-(\alpha t)^{-\beta})}.$$

Khan et al. (2008) studied the inverse Weibull distribution and ascertained its flexibility in modelling reliability data. They further discovered that the inverse Weibull (IW) distribution approaches different distributions when its shape parameter changes.

New families of distributions have been formulated by compounding the Poisson distribution with other continuous distributions to provide more flexibility for lifetime data. The zero truncated version of the Poisson distribution is defined as the zero-truncated binomial distribution for the Poisson distribution. Jones (2009) advocated for the Kumaraswamy distribution on the grounds that firstly, it had closed form solutions of the distribution and quantile functions; secondly, the simplicity of the formulas for the moments and thirdly, the fact that the distribution had a simple normalizing constant. Jones further contrasted the Kumaraswamy distribution with the beta distribution.

The motivation of this study arises from the advantages observed in the Kumaraswamy inverse Weibull distribution in modelling lifetime data as well as the flexibility in the hazard function which has numerous shapes such as the bathtub shape, increasing and decreasing shape. We propose and study a new class of distribution which inherits these important and desirable properties and contains several submodels with a large number of shapes.

Thesis Outline

The outline of this thesis is as follows: In chapter two, we present some basic information related to generalized distributions and Kumaraswamy generalized distributions. Properties of the (KIWP) distribution are derived in Chapter 3, including expansion of density, hazard function, monotonicity property, shapes, moments, reliability, mean deviations, Bonferroni and Lorenz curves. Measures of uncertainty such as Renyi entropy and Shannon entropy as well as Fisher information are presented. The mle maximum likelihood is used to estimate the parameters of the (KIWP) and related distributions. Finally, real data examples are discussed to illustrate the applicability of this class of models.

CHAPTER 2

THE KUMARASWAMY INVERSE WEIBULL POISSON DISTRIBUTION

General Class of Distribution

The probability distribution function (pdf) and corresponding cumulative distribution function (cdf) of the two-parameter inverse Weibull distribution are respectively given by

$$f(x;\alpha,\beta) = \alpha\beta x^{-\beta-1}\exp(-\alpha x^{-\beta}), \quad x>0, \alpha>0, \beta>0 \tag{2.1}$$

and

$$F(x;\alpha,\beta) = \exp(-\alpha x^{-\beta}), \quad x>0, \alpha>0, \beta>0. \tag{2.2}$$

Suppose that the random variable X has the inverse Weibull (IW) distribution where its pdf and cdf are given in equations 2.1 and 2.2. Given N, let $X_1, ..., X_N$ be independent and identically distributed random variables from the IW distribution. Let N be distributed according to the zero truncated Poisson distribution with pdf

$$P(N=n) = \frac{\theta^n e^{-\theta}}{n!(1-e^{-\theta})}, \quad n=1,2,...,\theta>0.$$

Let X=max$(Y_1, ..., Y_N)$, then the cdf of $X|N=n$ is given by

$$G_{X|N=n}(x) \;=\; \exp(-n\alpha x^{-\beta}) \quad x>0,\ \alpha>0,\ \beta>0,$$

which is the inverse Weibull distribution. The inverse Weibull-Poisson distribution denoted by IWP(α, β, θ) is defined by the marginal cdf of X, that is,

$$G_{IWP}(x; \alpha, \beta, \theta) \quad = \quad \frac{1 - \exp[\theta e^{-\alpha x^{-\beta}}]}{1 - e^{\theta}} \tag{2.3}$$

for $x > 0$, $\beta > 0$, $\theta > 0$. The graph of the distribution function for various values of the parameters α, β, θ, a and b is given in Figure 1.

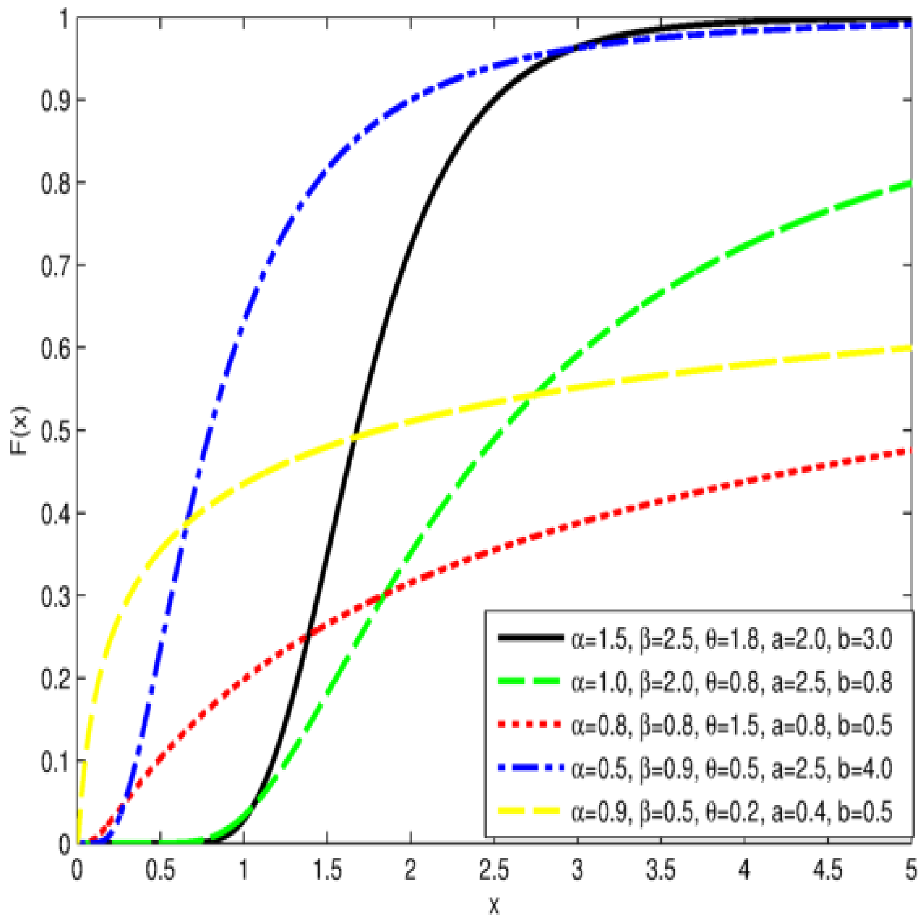

Figure 1. Plot of the cdf for different parameters.

The IWP density function is given by

$$g_{IWP}(x; \alpha, \beta, \theta) = \frac{\theta\alpha\beta x^{-\beta-1} e^{-\alpha x^{-\beta}} \exp[\theta e^{-\alpha x^{-\beta}}]}{e^{\theta} - 1} \tag{2.4}$$

for $x > 0, \beta > 0, \theta > 0$, where α is the scale parameter and β is the shape parameter.The graph of the density function for various values of the parameters α, β, θ, a and b is given in Figure 2.

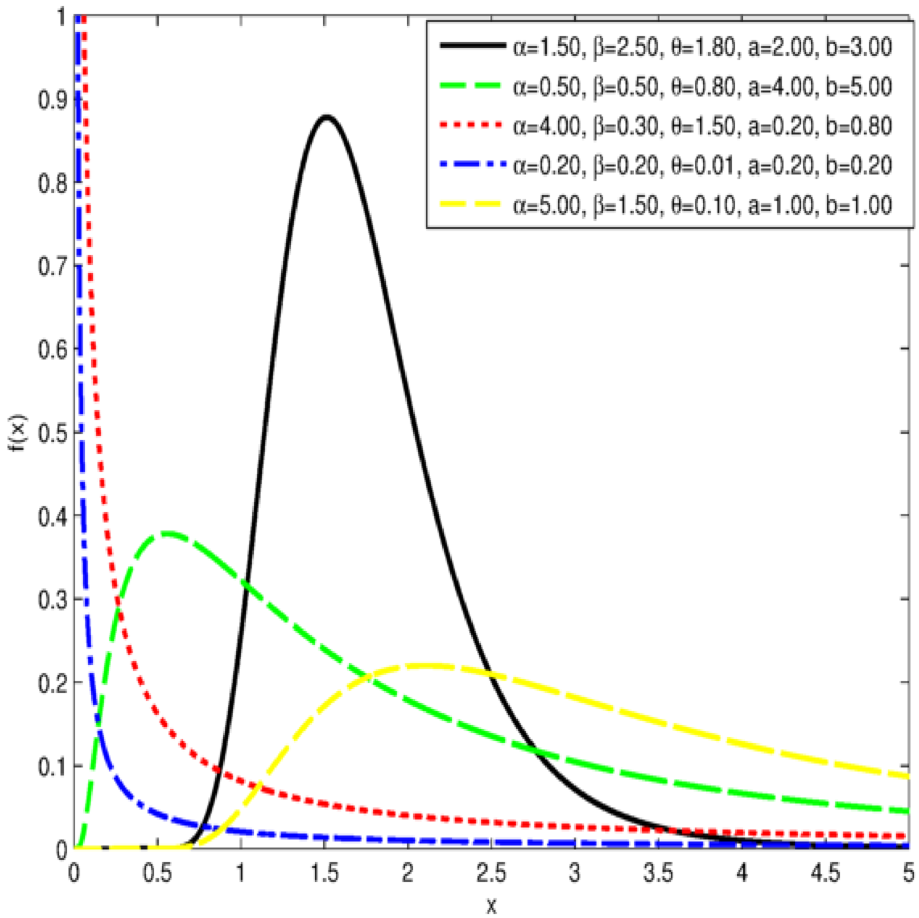

Figure 2. Plot of KIWP density for different parameters.

Expansion of the Density Function

The expansion of the pdf of KIWP distribution is presented in this section. For $b > 0$ a real non-integer, we use the series representation

$$(1-z)^{a-1} = \sum_{j=0}^{\infty} \binom{a-1}{j} (-1)^j z^j; a > 0, |z| < 1 \tag{2.5}$$

We thus have:

$$
\begin{aligned}
f_{KIWP}(x) &= abg(x)[G(x)]^{a-1}[1 - G(x)^{a}]^{b-1} \\
&= \sum_{j=0}^{\infty}(-1)^{j}\binom{b-1}{j} abg_{IWP}(x)[G_{IWP}(x)]^{aj+a-1}.
\end{aligned}
$$

$$
\begin{aligned}
f_{KIWP}(x) &= \sum_{j=0}^{\infty}\binom{b-1}{j}\frac{(-1)^{j}ab\alpha\beta\theta x^{-\beta-1}e^{-\alpha x^{-\beta}}\exp[\theta e^{-\alpha x^{-\beta}}]}{e^{\theta}-1} \\
&\times \left[\frac{1-\exp(\theta e^{-\alpha x^{-\beta}})}{1-e^{\theta}}\right]^{aj+a-1}
\end{aligned}
\tag{2.6}
$$

$$
\begin{aligned}
f_{KIWP}(x) &= \sum_{j=0}^{\infty}\binom{b-1}{j}\frac{(-1)^{aj+a+j-1}ab\alpha\beta\theta x^{-\beta-1}e^{-\alpha x^{-\beta}}\exp[\theta e^{-\alpha x^{-\beta}}]}{(e^{\theta}-1)^{aj+a}} \\
&\times \left[1-\exp(\theta e^{-\alpha x^{-\beta}})\right]^{aj+a-1} \\
&= \sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\binom{b-1}{j}\binom{aj+a-1}{k}\frac{(-1)^{aj+a+j+k-1}ab\alpha\beta\theta x^{-\beta-1}e^{-\alpha x^{-\beta}}}{(e^{\theta}-1)^{aj+a}} \\
&\times \exp\left[\theta(k+1)e^{-\alpha x^{-\beta}}\right].
\end{aligned}
$$

Notice that

$$
e^{t} = \sum_{m=0}^{\infty}\frac{t^{m}}{m!}.
$$

Applying the above identity to the last part of 2.6 yields:

$$
\begin{aligned}
f_{KIWP}(x) &= \sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\binom{b-1}{j}\binom{aj+a-1}{k}\frac{(-1)^{aj+a+j+k-1}}{(e^{\theta}-1)^{aj+a}}\frac{ab\theta^{m+1}(k+1)^m}{m!} \\
&\times \alpha\beta x^{-\beta-1}e^{-(m+1)\alpha x^{-\beta}} \\
&= \sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\binom{b-1}{j}\binom{aj+a-1}{k}\frac{(-1)^{aj+a+j+k-1}}{(e^{\theta}-1)^{aj+a}}\frac{ab\theta^{m+1}(k+1)^m}{(m+1)!} \\
&\times \alpha\beta(m+1)x^{-\beta-1}e^{-(m+1)\alpha x^{-\beta}} \\
&= \sum_{j,k,m=0}^{\infty}\omega_{j,k,m}(a,b,\theta)g(x;\alpha[m+1],\beta), \qquad (2.7)
\end{aligned}
$$

where

$$
\omega_{j,k,m}(a,b,\theta) = \binom{b-1}{j}\binom{aj+a-1}{k}\frac{(-1)^{aj+a+j+k-1}}{(e^{\theta}-1)^{aj+a}}\frac{ab\theta^{m+1}(k+1)^m}{(m+1)!} \qquad (2.8)
$$

are the weights and $g(x;\alpha(m+1),\beta)$ is the inverse Weibull pdf with scale parameter $\alpha(m+1)$ and shape parameter β. Consequently, the KIWP density can be written as a linear combination of inverse Weibull density functions. The mathematical properties of the KIWP follow directly from those of the IW distribution.

Monotonicity Properties

The monotonicity properties of the KIWP distribution are discussed in this section. Let

$$
V(x) = \left(\frac{1-\exp[\theta e^{(-\alpha x)^{-\beta}}]}{1-e^{\theta}}\right) \qquad (2.9)
$$

From equation (2.9) we can rewrite the KIWP pdf as

$$
\begin{aligned}
f_{KIWP}(x;\alpha,\beta,\theta,a,b) &= \frac{ab\alpha\beta\theta x^{-\beta-1}e^{-\alpha x^{-\beta}}[1-V(x)(1-e^{\theta})]}{e^{\theta}-1} \\
&\times V(x)^{a-1}\times(1-V(x)^a)^{b-1}
\end{aligned}
$$

for $x > 0$, $\alpha > 0$, $\beta > 0$, $\theta > 0$, $a > 0$, $b > 0$. It follows that

$$\begin{aligned} \log f_{KIWP}(x) &= \log(ab\alpha\beta\theta) - \log x^{-\beta-1} - \alpha x^{-\beta} + \log[1 - V(x) + V(x)e^{\theta}], \\ &\quad - \log(e^{\theta} - 1) + (a-1)\log V(x) + (b-1)\log(1 - V(x)^a) \end{aligned}$$

and

$$\begin{aligned} \frac{d \log f_{KIWP}(x)}{dx} &= -\frac{\beta+1}{x} + \frac{V'(x)(e^{\theta}-1)}{1 - V(x) + V(x)e^{\theta}} + (a-1)\frac{V'(x)}{V(x)} \\ &- a(b-1)\frac{V(x)^{a-1}V'(x)}{1 - V(x)^a}. \end{aligned}$$

Substituting $V'(x) = dV(x)/dx = \left(\frac{-1}{1-e^{\theta}}\right)\alpha\beta\theta x^{-\beta-1}e^{-\alpha x^{-\beta}}\exp[\theta e^{-\alpha x^{-\beta}}]$ into equation 2.10, we have

$$\begin{aligned} \frac{d \log f_{KIWP}(x)}{dx} &= -\frac{\beta+1}{x} + \beta\alpha x^{-\beta-1} + \frac{V'(x)(e^{\theta}-1)}{1 - V(x) + V(x)e^{\theta}} + (a-1)\frac{V'(x)}{V(x)}, \\ &\quad -a(b-1)\frac{V'(x)V(x)^{a-1}(x)}{1 - V(x)^a}. \end{aligned}$$

Taking the 2nd partial derivative gives:

$$\begin{aligned} \frac{d^2 \log f_{KIWP}(x)}{dx} &= \frac{\beta+1}{x^2} - \alpha\beta(\beta+1)x^{-\beta-2}, \\ &\quad + \frac{V''(x)(e^{\theta}-1)(1 - V(x) + V(x)e^{\theta}) - V(x)'(e^{\theta}-1)(V'(x)e^{\theta}-1)}{(1 - V(x) + V(x)e^{\theta})^2}, \\ &\quad +(a-1)\frac{V''(x)V(x) - (V'(x))^2}{V(x)^2}, \\ &\quad -a(b-1)\frac{(a-1)V(x)^{a-2}(V'(x))^2(1 - V(x)^a) - aV(x)^{a-1}(V'(x))^2V(x)^{a-1}}{(1 - V(x)^a)^2} \end{aligned}$$

Thus:

$$\frac{d^2 \log f_{KIWP}(x)}{dx} = \frac{\beta+1}{x^2} - \alpha\beta(\beta+1)x^{-\beta-2} + q(x)$$

where

$$\begin{aligned} q(x) &= \frac{V''(x)(e^\theta - 1)(1 - V(x) + V(x)e^\theta) - V'(x)(e^\theta - 1)(V(x)'e^\theta - 1)}{(1 - V(x) + V(x)e^\theta)^2}, \\ &+(a-1)\frac{V''(x)V(x) - (V'(x))^2}{V(x)^2}, \\ &-a(b-1)\frac{(a-1)V(x)^{a-2}(V'(x))^2(1 - V(x)^a) - aV(x)^{a-1}(V(x)')^2V(x)^{a-1}}{(1 - V(x)^a)^2}. \end{aligned}$$

Since $\alpha > 0$, $\beta > 0$, $\theta > 0$, $a > 0$ and $b > 0$, we have

$$V'(x) = \frac{dV(x)}{dx} = \left(\frac{1 - \exp[\theta e^{(-\alpha x)^{-\beta}}]}{1 - e^\theta}\right) \frac{-\alpha\beta\theta(\alpha x)^{-\beta-1}\exp[\theta e^{-\alpha x^{-\beta}}]}{1 - e^\theta},$$

for all $x > 0$. If $x \longrightarrow 0$, then

$$V(x) = \left(\frac{1 - \exp[\theta e^{(-\alpha x)^{-\beta}}]}{1 - e^\theta}\right) \longrightarrow 1.$$

If $x \longrightarrow \infty$, then

$$V(x) = \left(\frac{1 - exp[\theta e^{(-\alpha x)^{-\beta}}]}{1 - e^\theta}\right) \longrightarrow 0.$$

Since $\lim_{x\to+\infty} \exp[\theta e^{(-\alpha x)^{-\beta}}] = 1$ then $\lim_{x\to+\infty}(1 - \exp[\theta e^{(-\alpha x)^{-\beta}}]) = 0$. Thus $V(x)$ is monotonically increasing from 0 to 1.

Note that, since $0 < V(x) < 1$,$0 < V^{a-1}(x) < 1$, for all $a > 1$,

$0 < 1 - V^{a-1}(x) < 1$, for all $a > 0$ and $V'(x) > 0$, we have $0 < [1 - V^a(x)]^{b-1} < 1$.

for all $a > 1$, and $b > 1$. If $\alpha > 0$, $f_{KIWP}(x; \alpha, \beta, \theta, a, b)$ could attain a maximum, a minimum or a point of inflection according to whether

$$\frac{d^2 \log f_{KIWP}(x)}{dx^2} < 0, \quad \frac{d^2 \log f_{KIWP}(x)}{dx^2} > 0 \quad \text{or} \quad \frac{d^2 \log f_{KIWP}(x)}{dx^2} = 0.$$

Survival, Hazard and Reverse Hazard Functions

The hazard and reverse hazard functions for the KIWP distribution will be presented in this section. The survival function of the KIWP distribution is given by:

$$S(x) = \overline{F}(x) = 1 - F_{KIWP}(x) = \left[1 - \left(\frac{1-\exp[\theta e^{-(\alpha x)^{-\beta}}]}{1-e^{\theta}}\right)^{a}\right]^{b}.$$

The hazard and reverse hazard functions of the KIWP are given respectively by

$$\begin{aligned} h(x) &= \frac{f_{KIWP}(x;\alpha,\beta,\theta,a,b)}{1-F_{KIWP}(x;\alpha,\beta,\theta,a,b)} \\ &= \frac{ab\theta\alpha\beta x^{-\beta-1}e^{-(\alpha x)^{-\beta}}\exp[\theta e^{-(\alpha x)^{-\beta}}]}{e^{\theta}-1}\left[\frac{1-\exp[\theta e^{-(\alpha x)^{-\beta}}]}{1-e^{\theta}}\right]^{a-1} \\ &\times \left[1-\left(\frac{1-\exp[\theta e^{-(\alpha x)^{-\beta}}]}{1-e^{\theta}}\right)^{a}\right]^{-1} \\ &= \frac{ab\theta\alpha\beta x^{-\beta-1}e^{-(\alpha x)^{-\beta}}\exp[\theta e^{-(\alpha x)^{-\beta}}]\left(\exp[\theta e^{-(\alpha x)^{-\beta}}]-1\right)^{a-1}}{\left(e^{\theta}-1\right)^{a}-\left(\exp[\theta e^{-(\alpha x)^{-\beta}}]-1\right)^{a}}, \end{aligned}$$

and

$$\begin{aligned} \tau(x) &= \frac{f_{KIWP}(x;\beta,\theta,a,b)}{F_{KIWP}(x;\beta,\theta,a,b)} \\ &= \frac{ab\theta\alpha\beta x^{-\beta-1}e^{-(\alpha x)^{-\beta}}\exp[\theta e^{-(\alpha x)^{-\beta}}]}{e^{\theta}-1}\left(\frac{1-\exp[\theta e^{-(\alpha x)^{-\beta}}]}{1-e^{\theta}}\right)^{a-1} \\ &\times \left[1-\left(\frac{1-\exp[\theta e^{-(\alpha x)^{-\beta}}]}{1-e^{\theta}}\right)^{a}\right]^{b-1} \\ &\times \left\{1-\left[1-\left(\frac{1-\exp[\theta e^{-(\alpha x)^{-\beta}}]}{1-e^{\theta}}\right)^{a}\right]^{b}\right\}^{-1}, \end{aligned}$$

for $x > 0$, α, $\beta > 0$, $\theta > 0$, $a > 0$ and $b > 0$. The graph of the hazard function for various values of the parameters α, β, θ, a and b is given in Figure 3.

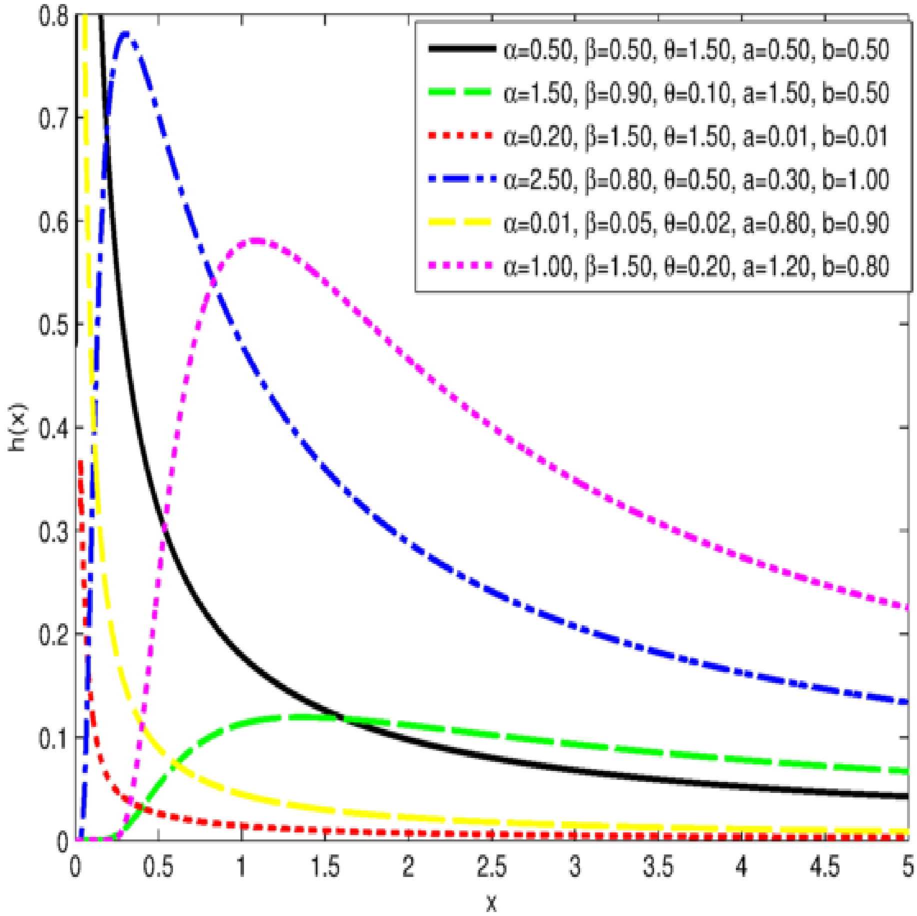

Figure 3. Plot of the hazard function for different parameter values.

The graph of the hazard function for different values of the parameters exhibits various shapes such as monotonically increasing, bathtub shape, increasing-decreasing, monotone decreasing and upside down bathtub shapes. This is an attractive feature that renders the KIWP distribution suitable for monotonic and non-monotonic hazard behaviors which are more likely to be encountered in real life situations.

Moments, Moment Generating Function and Conditional Moments

In this section, we present the moments, moment generating function and conditional moments of the KIWP distribution. Moments are necessary and important in any statistical analysis, especially in applications. They can be used to study the most important features and characteristics of a distribution such as tendency, dispersion, skewness and kurtosis.

Quantile Function

The quantile function of the KIWP distribution is obtained by solving the equation $F(x) = u$, where $0 < u < 1$. We therefore have

$$1 - \left[1 - \left(\frac{1 - \exp[\theta e^{-\alpha x^{-\beta}}]}{1 - e^{\theta}}\right)^{a}\right]^{b} \quad = \quad u.$$

Hence isolating the term $\exp(\theta e^{-\alpha x^{-\beta}})$ yields:

$$\frac{1 - \exp(\theta e^{-\alpha x^{-\beta}})}{1 - e^{\theta}} \quad = \quad \left[1 - (1 - u)^{1/b}\right]^{1/a}.$$

Thus,

$$\exp(\theta e^{-\alpha x^{-\beta}}) \quad = \quad 1 - (1 - e^{\theta})\left[1 - (1 - u)^{1/b}\right]^{1/a},$$

Taking natural logarithms both sides and dividing by θ yields:

$$e^{-\alpha x^{-\beta}} = \frac{1}{\theta} \log\left\{1 - (1 - e^{\theta})\left(1 - (1 - u)^{1/b}\right)^{1/a}\right\}$$

Thus,

$$-\alpha x^{-\beta} = \log\left[\frac{1}{\theta}\log\left\{1-(1-e^{\theta})\left(1-(1-u)^{1/b}\right)^{1/a}\right\}\right]$$

Dividing both sides by $-\alpha$

$$x^{-\beta} = \frac{-1}{\alpha}\log\left[\frac{1}{\theta}\log\left\{1-(1-e^{\theta})\left(1-(1-u)^{1/b}\right)^{1/a}\right\}\right].$$

The quantile function of the KIWP distribution is obtained by solving for x in the cdf to obtain

$$x = \left(\frac{-1}{\alpha}\log\left[\frac{1}{\theta}\log\left\{1-(1-e^{\theta})\left(1-(1-u)^{1/b}\right)^{1/a}\right\}\right]\right)^{-1/\beta}.$$

Moments

In this section, moments and related measures including coefficients of variation, skewness and kurtosis for the KIWP distribution are presented. A table of values for mean, standard deviation, coefficient of variation (CV), coefficient of skewness (CS) and coefficient of kurtosis (CK) is also presented. The r^{th} moment of a random variable X following the KIWP distribution, denoted by $E(X^r) = \mu_r'$ is given by

$$\begin{aligned} E(X^r) &= \int_0^{\infty} x^r f_{KIWP}(x;\alpha,\beta,\theta,a,b)dx \\ &= \sum_{j,k,m=0}^{\infty} \omega_{j,k,m}(a,b,\theta)\alpha\beta(m+1)\int_0^{\infty} x^{r-\beta-1}e^{-(m+1)\alpha x^{-\beta}}dx \\ &= \sum_{j,k,m=0}^{\infty} \omega_{j,k,m}(a,b,\theta)\int_0^{\infty} x^r e^{-(m+1)\alpha x^{-\beta}}\alpha\beta(m+1)x^{-\beta-1}dx. \end{aligned}$$

By letting $u = (m+1)\alpha x^{-\beta}$, we have $du = -\alpha\beta(m+1)x^{-\beta-1}dx$ and $x = u^{-1/\beta}[\alpha(m+1)]^{1/\beta}$. Thus

$$\begin{aligned}\int_0^\infty x^r e^{-(m+1)\alpha x^{-\beta}} \alpha\beta(m+1)x^{-\beta-1}dx &= [\alpha(m+1)]^{\frac{r}{\beta}} \int_0^\infty u^{(1-\frac{r}{\beta})-1} e^{-u} du \\ &= [\alpha(m+1)]^{r/\beta}\Gamma\left(1-\frac{r}{\beta}\right),\end{aligned}$$

where $\beta > r$. Hence, $E(X^r)$ is given by

$$\mu'_r = \sum_{j,k,m=0}^{\infty} \omega_{j,k,m}(a,b,\theta)[\alpha(m+1)]^{\frac{r}{\beta}}\Gamma\left(1-\frac{r}{\beta}\right), \tag{2.10}$$

where $\beta > r$ and $\omega_{j,k,m}(a,b,\theta)$ is defined in equation 2.8.The mean of the KIWP distribution is

$$\mu = \mu'_1 = \sum_{j,k,m=0}^{\infty} \omega_{j,k,m}(a,b,\theta)[\alpha(m+1)]^{1/\beta}\Gamma\left(1-\frac{1}{\beta}\right),$$

for $\beta > 1$. The variance, CV, CS, and CK are given by

$$\sigma^2 = \mu'_2 - \mu^2,$$

$$CV = \frac{\sigma}{\mu} = \frac{\sqrt{\mu'_2 - \mu^2}}{\mu} = \sqrt{\frac{\mu'_2}{\mu^2} - 1},$$

$$CS = \frac{E\left[(X-\mu)^3\right]}{\left[E(X-\mu)^2\right]^{3/2}} = \frac{\mu'_3 - 3\mu\mu'_2 + 2\mu^3}{(\mu'_2 - \mu^2)^{3/2}},$$

and

$$CK = \frac{E\left[(X-\mu)^4\right]}{\left[E(X-\mu)^2\right]^2} = \frac{\mu'_4 - 4\mu\mu'_3 + 6\mu^2\mu'_2 - 3\mu^4}{(\mu'_2 - \mu^2)^2},$$

respectively. Table 1 lists the first six moments of the KIWP distribution for selected values of the parameters obtained by fixing $\alpha = 1.5$, $\beta = 6.5$ and $\theta = 1.5$. Table 2 lists the first six moments of the KIWP distribution for selected values of the parameters obtained by fixing $a = 1.0$ and $b = 1.0$. These values can be determined numerically using R and MATLAB.

Table 1. *Moments of the KIWP Distribution for* $\alpha = 1.5, \beta = 6.5$ *and* $\theta = 1.5$.

μ'_s	$a = 1.0, b = 1.5$	$a = 1.0, b = 3.5$	$a = 1.0, b = 1.0$	$a = 2.5, b = 1.0$
μ'_1	1.189	1.057	1.292	1.502
μ'_2	1.459	1.132	1.767	2.380
μ'_3	1.856	1.229	2.606	4.054
μ'_4	2.465	1.353	4.304	7.703
μ'_5	3.453	1.512	8.731	17.910
μ'_6	5.187	1.716	33.028	77.371
SD	0.213	0.121	0.312	0.351
CV	0.179	0.115	0.242	0.234
CS	1.464	0.677	2.362	2.474
CK	8.050	3.931	17.971	19.174

Table 2. *Moments of the KIWP Distribution for* $a = 1.0$ *and* $b = 1.0$.

μ'_s	$\alpha = 0.8, \beta = 6.5$	$\alpha = 1.3, \beta = 7.5$	$\alpha = 2.0, \beta = 8.5$	$\alpha = 1.5, \beta = 7.0$
μ'_1	1.204	1.249	1.216	1.178
μ'_2	1.536	1.628	1.524	1.449
μ'_3	2.114	2.235	1.982	1.887
μ'_4	3.261	3.295	2.703	2.671
μ'_5	6.178	5.408	3.938	4.343
μ'_6	21.832	10.829	6.353	9.651
SD	0.293	0.258	0.212	0.247
CV	0.243	0.206	0.174	0.210
CS	2.325	2.038	1.967	2.405
CK	17.671	13.504	12.040	17.323

Moment Generating Function

The moment generating function of the KIWP distribution can be obtained from the r^{th} moment which is given as follows:

$$
\begin{aligned}
E(e^{tX}) &= \sum_{i=0}^{\infty} \frac{t^i}{i!} E(X^i) \\
&= \sum_{i=0}^{\infty} \sum_{j,k,m=0}^{\infty} \omega_{j,k,m}(a,b,\theta)[\alpha(m+1)]^{i/\beta} \Gamma\left(1-\frac{i}{\beta}\right), \qquad (2.11)
\end{aligned}
$$

$\beta > i$ where $\omega_{j,k,m}(a,b,\theta)$ is defined by equation 2.8

Conditional Moments

For income and lifetime distributions, it is of interest to obtain the conditional moments and mean residual life function. The r^{th} conditional moment for KIWP distribution is given by

$$
\begin{aligned}
E(X^r|X>t) &= \frac{1}{\overline{F}_{KIWP}(t)} \int_t^{\infty} x^r f_{KIWP}(x)dx \\
&= \frac{1}{\overline{F}_{KIWP}(t)} \sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty} \binom{b-1}{j}\binom{aj+a-1}{k} \frac{(-1)^{aj+a+j+k-1}}{(e^{\theta}-1)^{aj+a}} \\
&\times \frac{ab\theta^{m+1}(k+1)^m}{(m+1)!} \int_t^{\infty} \alpha\beta(m+1)x^{r-\beta-1}e^{-(m+1)\alpha x^{-\beta}}dx. \qquad (2.12)
\end{aligned}
$$

Considering the integral part, we thus have

$$
\begin{aligned}
&\int_t^{\infty} \alpha\beta(m+1)x^{r-\beta-1}e^{-(m+1)\alpha x^{-\beta}}dx \\
&= \int_t^{\infty} x^r e^{-(m+1)\alpha x^{-\beta}} \alpha\beta(m+1)x^{-\beta-1}dx.
\end{aligned}
$$

By letting $u=(m+1)\alpha x^{-\beta}$, we have $du=-\alpha\beta(m+1)x^{-\beta-1}dx$ and $x=u^{-1/\beta}[\alpha(m+1)]^{1/\beta}$. When $x=t$, $u=\alpha(m+1)t^{-\beta}$ and when $x=\infty$, $u=0$. We therefore have

$$\begin{aligned}
&\int_t^\infty x^r e^{-(m+1)\alpha x^{-\beta}}\alpha\beta(m+1)x^{-\beta-1}dx \\
&= -\int_{(m+1)\alpha t^{-\beta}}^{0}[\alpha(m+1)]^{\frac{r}{\beta}}u^{\frac{-r}{\beta}}e^{-u}du \\
&= [\alpha(m+1)]^{\frac{r}{\beta}}\int_0^{(m+1)\alpha t^{-\beta}}u^{\frac{-r}{\beta}}e^{-u}du \\
&= [\alpha(m+1)]^{\frac{r}{\beta}}\gamma\left(1-\frac{r}{\beta},\alpha(m+1)t^{-\beta}\right),
\end{aligned}$$

after applying the lower incomplete gamma function

$$\gamma(s,a)=\int_0^a t^{s-1}e^{-t}dt.$$

The r^{th} conditional moment for the KIWP distribution is given by

$$\begin{aligned}
E(X^r|X>t) &= \left[1-\left(\frac{1-\exp[\theta e^{-\alpha t^{-\beta}}]}{1-e^{\theta}}\right)^a\right]^{-b}\sum_{j,k,m=0}^{\infty}\omega_{j,k,m}(a,b,\theta) \\
&\times [\alpha(m+1)]^{\frac{r}{\beta}}\gamma\left(1-\frac{r}{\beta},\alpha(m+1)t^{-\beta}\right),
\end{aligned}$$

where $\omega_{j,k,m}(a,b,\theta)$ is defined by 2.8. The mean residual life function is $E(X^r|X>t)-t$.

Order Statistics

Order statistics are used to find the characteristics such as minimum and maximum time to failure of electronic components for example lightbulbs in reliability theory. This may be achieved by testing n lightbulbs and simultaneously putting them on test and collecting the time to failure as the successive failures occur. The observations may then be ordered as $X_1, X_2, ..., X_n$ whereby X_1 denotes the minimum time to failure and X_n denotes the maximum time to failure. The trials are independent and identically distributed. The pdf of the k^{th} order statistic from the KIWP distribution is

$$f_{k:n}(x) = \frac{n! f_{KIWP}(x)}{(k-1)!(n-k)!}[F_{KIWP}(x)]^{k-1}[1 - F_{KIWP}(x)]^{n-k}.$$

Using the identity

$$(1-z)^{a-1} = \sum_{p=0}^{\infty}(-1)^p \binom{a-1}{p} z^p,$$

we have

$$\begin{aligned}
& \frac{n! f_{KIWP}(x)}{(k-1)!(n-k)!}\sum_{p=0}^{\infty}(-1)^p \binom{n-k}{p}[F_{KIWP}(x)]^{p+k-1} \\
= & \frac{n! f_{KIWP}(x)}{(k-1)!(n-k)!}\sum_{p=0}^{\infty}(-1)^p \binom{n-k}{p} \\
\times & \left\{1 - \left[1 - \left(\frac{1-\exp[\theta e^{-\alpha x^{-\beta}}]}{1-e^{\theta}}\right)^a\right]^b\right\}^{p+k-1} \\
= & \frac{n! f_{KIWP}(x)}{(k-1)!(n-k)!}\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}(-1)^{p+q} \binom{n-k}{p}\binom{p+k-1}{q} \\
\times & \left[1 - \left(\frac{1-\exp[\theta e^{-\alpha x^{-\beta}}]}{1-e^{\theta}}\right)^a\right]^{bq}.
\end{aligned}$$

Substituting the KIWP pdf into the above equation gives

$$
\begin{aligned}
= & \frac{n!}{(k-1)!(n-k)!}\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}(-1)^{p+q}\binom{n-k}{p}\binom{p+k-1}{q} \\
\times & \frac{ab\theta\alpha\beta x^{-\beta-1}e^{-\alpha x^{-\beta}}\exp[\theta e^{-\alpha x^{-\beta}}]}{e^{\theta}-1}\left[\frac{1-\exp[\theta e^{-\alpha x^{-\beta}}]}{1-e^{\theta}}\right]^{a-1} \\
\times & \left[1-\left(\frac{1-\exp[\theta e^{-\alpha x^{-\beta}}]}{1-e^{\theta}}\right)^{a}\right]^{bq+b-1}
\end{aligned}
$$

$$
\begin{aligned}
= & \frac{n!}{(k-1)!(n-k)!}\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\sum_{s=0}^{\infty}(-1)^{p+q+s}\binom{n-k}{p}\binom{p+k-1}{q} \\
\times & \binom{bq+b-1}{s}\frac{ab\theta\alpha\beta x^{-\beta-1}e^{-\alpha x^{-\beta}}\exp[\theta e^{-\alpha x^{-\beta}}]}{e^{\theta}-1} \\
\times & \left[\frac{1-\exp[\theta e^{-\alpha x^{-\beta}}]}{1-e^{\theta}}\right]^{as+a-1} \\
= & \frac{n!}{(k-1)!(n-k)!}\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\sum_{s=0}^{\infty}(-1)^{p+q+s+a+as-1} \\
\times & \binom{n-k}{p}\binom{p+k-1}{q}\binom{bq+b-1}{s} \\
\times & \frac{ab\theta\alpha\beta x^{-\beta-1}e^{-\alpha x^{-\beta}}\exp[\theta e^{-\alpha x^{-\beta}}]}{(e^{\theta}-1)^{as+a}}\left[1-\exp(\theta e^{-\alpha x^{-\beta}})\right]^{as+a-1} \\
= & \frac{n!}{(k-1)!(n-k)!}\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\sum_{s=0}^{\infty}\sum_{i=0}^{\infty}(-1)^{p+q+s+a+as+i-1} \\
\times & \binom{n-k}{p}\binom{p+k-1}{q}\binom{bq+b-1}{s}\binom{as+a-1}{i} \\
\times & \frac{ab\theta\alpha\beta x^{-\beta-1}e^{-\alpha x^{-\beta}}\exp[\theta(i+1)e^{-\alpha x^{-\beta}}]}{(e^{\theta}-1)^{as+a}}.
\end{aligned}
$$

Using the expansion

$$e^{x}=\sum_{t=0}^{\infty}\frac{x^{t}}{t!},$$

$$
\begin{aligned}
e^{-\alpha x^{-\beta}} \exp[\theta(i+1)e^{-\alpha x^{-\beta}}] &= \sum_{j=0}^{\infty} \frac{[\theta(i+1)]^j e^{-\alpha(j+1)x^{-\beta}}}{j!} \\
&= \sum_{j=0}^{\infty} \frac{[\theta(i+1)]^j (j+1) e^{-\alpha(j+1)x^{-\beta}}}{j!(j+1)}.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
f_{k:n}(x) &= \frac{n!}{(k-1)!(n-k)!} \sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\sum_{s=0}^{\infty}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} (-1)^{p+q+s+a+as+i-1} \\
&\times \binom{n-k}{p}\binom{p+k-1}{q}\binom{bq+b-1}{s}\binom{as+a-1}{i} \\
&\times \frac{ab\theta^{j+1}(i+1)^j}{(j+1)!(e^{\theta}-1)^{as+a}} \alpha\beta(j+1)x^{-\beta-1}e^{-\alpha(j+1)x^{-\beta}} \\
&= \sum_{p,q,s,i,j=0}^{\infty} Q(p,q,s,i,j) f_{IW}(x;\alpha(j+1),\beta),
\end{aligned}
$$

where

$$
\begin{aligned}
Q(p,q,s,i,j) &= \frac{n!}{(k-1)!(n-k)!} \frac{(-1)^{p+q+s+a+as+i-1} ab\theta^{j+1}(i+1)^j}{(j+1)!(e^{\theta}-1)^{as+a}} \\
&\times \binom{n-k}{p}\binom{p+k-1}{q}\binom{bq+b-1}{s}\binom{as+a-1}{i}.
\end{aligned}
$$

Thus, the pdf of the k^{th} order statistic from the KIWP distribution is clearly a linear combination of inverse Weibull pdfs with parameters $\alpha(j+1)$, $\beta > 0$. The r^{th} moment of the distribution of the k^{th} order statistic is given by

$$
\begin{aligned}
E(X_{k:n}^r) &= \sum_{p,q,s,i,j=0}^{\infty} Q(p,q,s,i,j) \int_0^{\infty} x^r f_{IW}(x;\alpha(j+1),\beta)dx \\
&= \sum_{p,q,s,i,j=0}^{\infty} Q(p,q,s,i,j)[\alpha(j+1)]^{\frac{r}{\beta}} \Gamma\left(1-\frac{r}{\beta}\right),
\end{aligned}
$$

where $\beta > r$.

Sub-Models of the KIWP distribution

In this section, some sub-models of the KIWP distribution for selected values of the parameters α, β, θ, a and b are presented.

(1) $\boldsymbol{a = b = 1}$

When $a = b = 1$, we obtain the inverse Weibull Poisson (IWP) distribution whose cumulaltive distribution function (cdf) and probability density function (pdf) are respectively given by

$$\begin{aligned} F(x, \alpha, \beta) &= \exp(-\alpha x^{-\beta}), x > 0, \alpha > 0, \beta > 0, \\ f(x, \alpha, \beta) &= \alpha\beta x^{-\beta-1} e^{-\alpha} x^{-\beta}. \end{aligned}$$

(2) $\boldsymbol{b = 1}$

When $b = 1$, we obtain the exponentiated inverse Weibull Poisson (EIWP) distribution which belongs to the resilience parameter family and whose cdf is given by

$$F(x; \alpha, \beta, \theta, a) = \left[\frac{1 - \exp(\theta e^{-\alpha x^{-\beta}})}{1 - e^{\theta}}\right]^{a},$$

with corresponding pdf

$$f(x; \alpha, \beta, \theta, a) = \frac{ab\theta\alpha\beta x^{-\beta-1} e^{-\alpha x^{-\beta}} \exp(\theta e^{-\alpha x^{-\beta}})}{e^{\theta} - 1} \left[\frac{1 - \exp(\theta e^{-\alpha x^{-\beta}})}{1 - e^{\theta}}\right]^{a-1},$$

for $x > 0, \alpha > 0, \beta > 0, \theta, a > 0$.

(3) $\boldsymbol{a = 1}$

When $a = 1$, we obtain the new lifetime distribution belonging to the frailty parameter family with cdf

$$1-\left[1-\left(\frac{1-\exp[\theta e^{-\alpha x^{-\beta}}]}{1-e^{\theta}}\right)\right]^{b},$$

and pdf

$$f(x) \quad = \quad \frac{b\theta\alpha\beta x^{-\beta-1}e^{-\alpha x^{-\beta}}\exp[\theta e^{-\alpha x^{-\beta}}]}{e^{\theta}-1}\left[1-\left(\frac{1-\exp[\theta e^{-\alpha x^{-\beta}}]}{1-e^{\theta}}\right)\right]^{b-1}$$

for $x > 0, \alpha, \beta > 0, \theta > 0, a > 0, b > 0$.

(4) $\boldsymbol{\beta = 2}$

When $\beta = 2$, we obtain the Kumaraswamy inverse Rayleigh Poisson (KIRP) distribution whose cdf is

$$F(x;\alpha,\theta,a,b) = 1-\left[1-\left(\frac{1-\exp[\theta e^{-\alpha x^{-2}}]}{1-e^{\theta}}\right)^{a}\right]^{b},$$

with corresponding pdf

$$\begin{aligned} f(x) \quad &= \quad \frac{2ab\theta\alpha x^{-3}e^{-\alpha x^{-2}}\exp[\theta e^{-\alpha x^{-2}}]}{e^{\theta}-1}\left[\frac{1-\exp[\theta e^{-\alpha x^{-2}}]}{1-e^{\theta}}\right]^{a-1} \\ &\times \quad \left[1-\left(\frac{1-\exp[\theta e^{-\alpha x^{-2}}]}{1-e^{\theta}}\right)^{a}\right]^{b-1}, \end{aligned}$$

for $x > 0, \alpha > 0, \theta, a > 0, b > 0$.

(5) $\boldsymbol{\beta = 1}$

When $\beta = 1$, we obtain the Kumaraswamy inverse exponential Poisson (KIEP) distribution whose cdf is

$$F(x;\alpha,\theta,a,b) = 1 - \left[1 - \left(\frac{1-\exp[\theta e^{-\alpha x^{-1}}]}{1-e^{\theta}}\right)^{a}\right]^{b}$$

with corresponding pdf

$$\begin{aligned} f(x) &= \frac{ab\theta\alpha x^{-2}e^{-\alpha x^{-1}}\exp[\theta e^{-\alpha x^{-1}}]}{e^{\theta}-1}\left[\frac{1-\exp[\theta e^{-\alpha x^{-1}}]}{1-e^{\theta}}\right]^{a-1} \\ &\times \left[1 - \left(\frac{1-\exp[\theta e^{-\alpha x^{-1}}]}{1-e^{\theta}}\right)^{a}\right]^{b-1}, \end{aligned}$$

for $x > 0, \alpha > 0, \theta, a > 0, b > 0$.

(6) $\boldsymbol{\alpha = 1}$

When $\alpha = 1$, we obtain the Kumaraswamy Frechet Poisson (KFP) distribution whose cdf is

$$1 - \left[1 - \left(\frac{1-\exp[\theta e^{-x^{-\beta}}]}{1-e^{\theta}}\right)^{a}\right]^{b}$$

with corresponding pdf

$$\begin{aligned} f_{KFP}(x) &= \frac{ab\theta\beta x^{-\beta-1}e^{-x^{-\beta}}\exp[\theta e^{-x^{-\beta}}]}{e^{\theta}-1}\left[\frac{1-\exp[\theta e^{-x^{-\beta}}]}{1-e^{\theta}}\right]^{a-1} \\ &\times \left[1 - \left(\frac{1-\exp[\theta e^{-x^{-\beta}}]}{1-e^{\theta}}\right)^{a}\right]^{b-1}, \end{aligned}$$

for $x > 0, \beta > 0, \theta > 0, a > 0, b > 0$.

(7) $\boldsymbol{\alpha = b = 1}$

When $\alpha = b = 1$, we get the exponentiated Frechet Poisson (EFP) distribution whose cdf is

$$F_{EFP}(x;a,\beta,\theta) = \left(\frac{1-\exp[\theta e^{-x^{-\beta}}]}{1-e^{\theta}}\right)^{a}.$$

The corresponding pdf is

$$\frac{a\theta\beta x^{-\beta-1}e^{-x^{-\beta}}\exp[\theta e^{-x}]}{e^{\theta}-1}\left[\frac{1-\exp[\theta e^{-x^{-\beta}}]}{1-e^{\theta}}\right]^{a-1} \tag{2.13}$$

for $x > 0, a > 0, \beta, \theta > 0$.

(8) $\boldsymbol{\alpha = a = b = 1}$

When $\alpha = a = b = 1$, we get the Frechet Poisson (FP) distribution whose cdf is

$$F_{FP}(x; a, \beta, \theta) = \frac{1-\exp[\theta e^{-x^{-\beta}}]}{1-e^{\theta}}.$$

The corresponding pdf is

$$f_{FP}(x) = \frac{\theta\alpha\beta x^{-\beta-1}e^{-x^{-\beta}}\exp[\theta e^{-\alpha x}]}{e^{\theta}-1}$$

for $x > 0, \theta > 0$.

(9) $\boldsymbol{\beta = b = 1}$

When $\beta = b = 1$, we get the exponentiated inverse exponential Poisson (EIEP) distribution whose cdf is

$$F_{EIEP}(x; a, \beta, \theta) = \left(\frac{1-\exp[\theta e^{-x^{-1}}]}{1-e^{\theta}}\right)^{a}.$$

The corresponding pdf is

$$f_{EIEP}(x) \quad = \quad \frac{a\theta\alpha x^{-2}e^{-\alpha x^{-1}}\exp[\theta e^{-\alpha x}]}{e^{\theta}-1}\left[\frac{1-\exp[\theta e^{-\alpha x^{-1}}]}{1-e^{\theta}}\right]^{a-1}$$

for $x > 0, a > 0, \beta > 0, \theta > 0.$

(10) $\boldsymbol{\beta = a = b = 1}$

When $\beta = a = b = 1$, we get the inverse exponential Poisson (IEP) distribution whose cdf is

$$F_{EIRP}(x; a, \beta, \theta) = \frac{1-\exp[\theta e^{-x^{-1}}]}{1-e^{\theta}}.$$

The corresponding pdf is

$$f_{EIRP}(x) \quad = \quad \frac{\theta\alpha x^{-2}e^{-\alpha x^{-1}}\exp[\theta e^{-\alpha x}]}{e^{\theta}-1}$$

.

for $x > 0, \theta > 0.$

(11) $\boldsymbol{\beta = 2, b = 1}$

When $\beta = 2, b = 1$, we get the exponentiated inverse Rayleigh Poisson (EIRP) distribution whose cdf is

$$F_{EIRP}(x; a, \beta, \theta) = \left(\frac{1-\exp[\theta e^{-x^{-2}}]}{1-e^{\theta}}\right)^{a}.$$

The corresponding pdf is

$$f_{EIRP}(x) \quad = \quad \frac{2a\theta\alpha x^{-3}e^{-\alpha x^{-2}}\exp[\theta e^{-\alpha x}]}{e^{\theta}-1}\left[\frac{1-\exp[\theta e^{-\alpha x^{-2}}]}{1-e^{\theta}}\right]^{a-1}$$

for $x > 0, a > 0, \alpha > 0, \theta > 0$.

(12) $\boldsymbol{\beta = 2, a = b = 1}$

When $\beta = 2, a = b = 1$, we get the inverse Rayleigh Poisson (IRP) distribution whose cdf is

$$F_{IRP}(x;\alpha,\theta) = \frac{1-\exp[\theta e^{-\alpha x^{-2}}]}{1-e^{\theta}}.$$

The corresponding pdf is

$$f_{IRP}(x) \quad = \quad \frac{2\theta\alpha x^{-3}e^{-\alpha x^{-2}}\exp[\theta e^{-\alpha x}]}{e^{\theta}-1}$$

for $x > 0, a > 0, \alpha > 0, \theta > 0$.

(13) $\boldsymbol{a = b = 1}$ and $\boldsymbol{\theta \to 0^+}$, we get the (IW) inverse Weibull distribution.

(14) $\boldsymbol{b = 1}$ and $\boldsymbol{\theta \to 0^+}$, we get the (EIW) exponentiated inverse Weibull distribution.

(15) $\boldsymbol{\beta = 2}$ and $\boldsymbol{\theta \to 0^+}$, we get the (KIR) Kum-inverse Rayleigh distribution.

(16) $\boldsymbol{\beta = 1}$ and $\boldsymbol{\theta \to 0^+}$, we get the (KIE) Kum-inverse exponential distribution.

(17) $\boldsymbol{\alpha = 1}$ and $\boldsymbol{\theta \to 0^+}$, we get the (KF) Kum-Frechet distribution.

(18) $\boldsymbol{\alpha = b = 1}$ and $\boldsymbol{\theta \to 0^+}$, we get the (EF) exponentiated-Frechet distribution.

(19) $\boldsymbol{\beta = b = 1}$ and $\boldsymbol{\theta \to 0^+}$, we get the (EIE) exponentiated-inverse exponential distribution.

(20) $\boldsymbol{\beta = a = b = 1}$ and $\boldsymbol{\theta \to 0^+}$, we get the (IE) inverse exponential distribution.

(21) $\boldsymbol{\beta = 2, b = 1}$ and $\boldsymbol{\theta \to 0^+}$, we get the (EIR) exponentiated inverse Rayleigh distribution.

(22) $\boldsymbol{\beta = 2, a = b = 1}$ and $\boldsymbol{\theta \to 0^+}$, we get the (IR) inverse Rayleigh distribution.

Mean and Median Deviations

The amount of dispersion in a population can be measured by the totality of deviations from the mean and the median. If X has the KIWP distribution, we can derive the mean deviation about the mean $\mu = E(X)$ and the mean deviation about the median $M = Median(X) = F^{-1}(1/2)$ from

$$\begin{aligned}
\delta_1 &= \int_0^\infty |x-\mu| f_{KIWP}(x)dx \\
&= \mu \int_0^\mu f(x)dx - \int_0^\mu xf(x)dx + \int_\mu^\infty xf(x)dx - \mu \int_\mu^\infty f(x)dx \\
&= \mu F(\mu) - \int_0^\infty xf(x)dx + 2\int_\mu^\infty xf(x)dx - \mu\left[1-F(\mu)\right] \\
&= 2\mu F(\mu) - 2\mu + 2\int_\mu^\infty xf(x)dx \\
&= 2\mu F(\mu) - 2\mu + 2\sum_{j,k,m=0}^{\infty} \omega_{j,k,m}(a,b,\theta)[\alpha(m+1)]^{\frac{1}{\beta}} \gamma\left(1-\frac{1}{\beta}, \alpha(m+1)\mu^{-\beta}\right),
\end{aligned} \tag{2.14}$$

and

$$\begin{aligned}
\delta_2 &= \int_0^\infty |x-M| f_{KIWP}(x)dx \\
&= \int_0^M (M-x)f(x)dx + \int_M^\infty (x-M)f(x)dx \\
&= M\int_0^M f(x)dx - \int_0^M xf(x)dx + \int_M^\infty xf(x)dx - M\int_M^\infty f(x)dx \\
&= MF(M) - \int_0^\infty xf(x)dx + 2\int_M^\infty xf(x)dx - M\left[1-F(M)\right] \\
&= 2MF(M) - M - \mu + 2\int_M^\infty xf(x)dx \\
&= 2M\frac{1}{2} - M - \mu + 2\int_M^\infty xf(x)dx \\
&= -\mu + 2\int_M^\infty xf(x)dx \\
&= -\mu + 2\sum_{j,k,m=0}^{\infty} \omega_{j,k,m}(a,b,\theta)[\alpha(m+1)]^{\frac{1}{\beta}}\gamma\left(1-\frac{1}{\beta}, \alpha(m+1)M^{-\beta}\right),
\end{aligned}$$

by using equation 2.14.

Bonferroni and Lorenz curves

Lorenz curves are income inequality measures that are also useful and applicable to other areas including reliability, demography, medicine and insurance. Lorenz and Bonferroni curves are given by

$$L(F(x)) = \frac{\int_0^x tf(t)dt}{E(X)}, \quad \text{and} \quad B(F(x)) = \frac{L(F(x))}{F(x)},$$

or

$$L(p) = \frac{1}{\mu}\int_0^q tf(t)dt, \quad \text{and} \quad B(p) = \frac{1}{p\mu}\int_0^q tf(t)dt,$$

respectively, where $q = F^{-1}(p)$. Now using equation 2.14, we can re-write the Lorenz and Bonferroni curves as

$$\begin{aligned}
B(p) &= \frac{1}{p\mu}\int_0^q tf(t)dt \\
&= \frac{1}{p\mu}\left[\int_0^\infty xf(x)dx - \int_q^\infty xf(x)dx\right] \\
&= \frac{1}{p\mu}\left[\mu - \sum_{j,k,m=0}^{\infty} \omega_{j,k,m}(a,b,\theta)[\alpha(m+1)]^{\frac{1}{\beta}}\gamma\left(1-\frac{1}{\beta}, \alpha(m+1)q^{-\beta}\right)\right].
\end{aligned}$$

and

$$\begin{aligned}
L(p) &= \frac{1}{\mu}\int_0^q tf(t)dt \\
&= \frac{1}{\mu}\left[\int_0^\infty xf(x)dx - \int_q^\infty xf(x)dx\right] \\
&= \frac{1}{\mu}\left[\mu - \sum_{j,k,m=0}^{\infty} \omega_{j,k,m}(a,b,\theta)[\alpha(m+1)]^{\frac{1}{\beta}}\gamma\left(1-\frac{1}{\beta}, \alpha(m+1)q^{-\beta}\right)\right],
\end{aligned}$$

where $\omega_{j,k,m}(a,b,\theta)$ is defined in equation 2.8.

CHAPTER 3

MEASURES OF UNCERTAINTY

Rényi Entropy

In this section we present some measures of uncertainty. The concept of entropy plays a vital role in information theory. The entropy of a random variable is defined in terms of its probability distribution and can be shown to be a good measure of randomness or uncertainty. Rényi entropy is an extension of Shannon entropy. Rényi entropy is defined to be

$$I_R(v) = \frac{1}{1-v}\log\left(\int_0^\infty [f_{KIWP}(x;a,b,\alpha,\beta,\theta)]^v dx\right), \quad v \neq 1, v > 0.$$

Rényi entropy tends to Shannon entropy as $v \to 1$. Notice that

$$\begin{aligned}
[f(x)]^v &= \left[\frac{ab\alpha\beta\theta}{e^\theta - 1}\right]^v x^{-\beta v - v} e^{-\alpha v x^{-\beta}} \exp[\theta v e^{-\alpha x^{-\beta}}]\left[\frac{1-\exp[\theta e^{-\alpha x^{-\beta}}]}{1-e^\theta}\right]^{av-v} \\
&\times \left[1 - \left(\frac{1-\exp[\theta e^{-\alpha x^{-\beta}}]}{1-e^\theta}\right)^a\right]^{bv-v}.
\end{aligned}$$

We thus have

$$\begin{aligned}
&\left[1 - \left(\frac{1-\exp[\theta e^{-\alpha x^{-\beta}}]}{1-e^\theta}\right)^a\right]^{bv-v}\left[\frac{1-\exp[\theta e^{-\alpha x^{-\beta}}]}{1-e^\theta}\right]^{av-v} \\
&= \sum_{i=0}^\infty \binom{bv-v}{i}(-1)^i\left[\frac{1-\exp[\theta e^{-\alpha x^{-\beta}}]}{1-e^\theta}\right]^{ai+av-v} \\
&= \sum_{i=0}^\infty \binom{bv-v}{i}\frac{(-1)^{i+ai+av-v}}{(e^\theta-1)^{ai+av-v}}\left[1-\exp(\theta e^{-\alpha x^{-\beta}})\right]^{ai+av-v} \\
&= \sum_{i=0}^\infty\sum_{j=0}^\infty \binom{bv-v}{i}\binom{ai+av-v}{j}\frac{(-1)^{i+j+ai+av-v}}{(e^\theta-1)^{ai+av-v}}\exp(\theta j e^{-\alpha x^{-\beta}}).
\end{aligned}$$

Thus

$$[f(x)]^v = \left[\frac{ab\alpha\beta\theta}{(e^\theta-1)^a}\right]^v \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\binom{bv-v}{i}\binom{ai+av-v}{j}\frac{(-1)^{i+j+ai+av-v}}{(e^\theta-1)^{ai}} \times x^{-\beta v-v}\exp\left(\theta(j+v)e^{-\alpha x^{-\beta}}\right)e^{-\alpha v x^{-\beta}}.$$

Using series expansions, we obtain:

$$[f(x)]^v = \left[\frac{ab\alpha\beta\theta}{(e^\theta-1)^a}\right]^v \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\binom{bv-v}{i}\binom{ai+av-v}{j}\frac{(-1)^{i+j+ai+av-v}\theta^k(j+v)^k}{k!(e^\theta-1)^{ai}} \times x^{-\beta v-v}e^{-\alpha(v+k)x^{-\beta}}.$$

Let $u=\alpha(v+k)x^{-\beta} \Rightarrow du=-\alpha\beta(v+k)x^{-\beta-1}dx$, and $x=[\alpha(v+k)]^{\frac{1}{\beta}}u^{\frac{-1}{\beta}}$. We have

$$\begin{aligned}
\int_0^\infty x^{-\beta v-v}e^{-\alpha(v+k)x^{-\beta}}dx &= \frac{1}{\alpha\beta(v+k)}\int_0^\infty x^{-\beta v+\beta-v+1}e^{-\alpha(v+k)x^{-\beta}}\alpha\beta(v+k)x^{-\beta-1}dx \\
&= \frac{[\alpha(v+k)]^{1-v+\frac{1-v}{\beta}}}{\alpha\beta(v+k)}\int_0^\infty u^{v-1+\frac{v-1}{\beta}}du \\
&= \frac{[\alpha(v+k)]^{\frac{1-v-v\beta}{\beta}}}{\beta}\Gamma\left(\frac{v\beta+v-1}{\beta}\right).
\end{aligned}$$

Therefore the Rényi entropy for the KIWP distribution is given by

$$I_R(v) = \frac{1}{1-v}\log\left\{\left[\frac{ab\alpha\beta\theta}{(e^\theta-1)^a}\right]^v \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\binom{bv-v}{i}\binom{ai+av-v}{j} \times \frac{(-1)^{i+j+ai+av-v}\theta^k(j+v)^k}{k!(e^\theta-1)^{ai}}\frac{[\alpha(v+k)]^{\frac{1-v-v\beta}{\beta}}}{\beta} \times \Gamma\left(\frac{v\beta+v-1}{\beta}\right)\right\}.$$

Shannon Entropy

Shannon entropy is a crucial concept in information theory. It measures the uncertainty associated with a random variable or the expected value of information encoded in a message in information theory. Shannon entropy is defined as
$H\left[f_{KIWP}\right] = E\left[-\log(f_{_{KIWP}}(X;\alpha,\beta,\theta,a,b))\right]$.
Thus we have

$$\begin{aligned} H\left[f_{KIWP}(X)\right] &= \log\left[\frac{e^{\theta}-1}{ab\alpha\beta\theta}\right] + (\beta+1)E\left[\log(X)\right] + \alpha E\left[x^{-\beta}\right] - \theta E\left[e^{-\alpha x^{-\beta}}\right] \\ &- (a-1)E\left[\log\left\{\frac{1-\exp(\theta e^{-\alpha x^{-\beta}})}{1-e^{\theta}}\right\}\right] \\ &- (b-1)E\left[\log\left\{\left[1-\left(\frac{1-\exp[\theta e^{-\alpha x^{-\beta}}]}{1-e^{\theta}}\right)^{a}\right]\right\}\right]. \end{aligned}$$

Note that

$$\log(x+1) = \sum_{q=1}^{\infty}\frac{(-1)^{q+1}x^{q}}{q}, -1 < x \leq 1.$$

$$\begin{aligned} E\left[\log\left\{\frac{1-\exp(\theta e^{-\alpha x^{-\beta}})}{1-e^{\theta}}\right\}\right] &= -\log(1-e^{\theta}) - \sum_{q=1}^{\infty}\frac{E\left[\exp(\theta q e^{-\alpha x^{-\beta}})\right]}{q} \\ &= -\log(1-e^{\theta}) - \sum_{q=1}^{\infty}\sum_{s=0}^{\infty}\frac{(q\theta)^{s}E\left[e^{-\alpha s x^{-\beta}}\right]}{qs!} \\ &= -\log(1-e^{\theta}) \\ &- \sum_{q=1}^{\infty}\sum_{s=0}^{\infty}\sum_{u=0}^{\infty}\frac{(-1)^{u}\theta^{s}q^{s-1}(\alpha s)^{u}}{s!u!}E[X^{-\beta u}]. \end{aligned}$$

$$E\left[\log\left\{\left[1-\left(\frac{1-\exp[\theta e^{-\alpha x^{-\beta}}]}{1-e^{\theta}}\right)^{a}\right]\right\}\right] = -\sum_{b=1}^{\infty}\frac{1}{b}E\left[\left(\frac{1-\exp(\theta e^{-\alpha x^{-\beta}})}{1-e^{\theta}}\right)^{ab}\right]$$
$$= -\sum_{b=1}^{\infty}\frac{(-1)^{ab}E\left\{\left[1-\exp(\theta e^{-\alpha x^{-\beta}})\right]^{ab}\right\}}{b(e^{\theta}-1)^{ab}}.$$

Using the identity

$$(1-z)^{a-1}=\sum_{i=0}^{\infty}\binom{a-1}{i}(-1)^{i},$$

we have

$$-\sum_{b=1}^{\infty}\frac{(-1)^{ab}[1-\exp(\theta e^{-\alpha x^{-\beta}})]^{ab}}{b(e^{\theta}-1)^{ab}} = -\sum_{b=1}^{\infty}\sum_{c=0}^{\infty}\binom{ab}{c}\frac{(-1)^{ab+c}}{b(e^{\theta}-1)^{ab}}E\left[\exp(\theta e^{-\alpha x^{-\beta}})\right]$$
$$= -\sum_{b=1}^{\infty}\sum_{c=0}^{\infty}\sum_{d=0}^{\infty}\sum_{g=0}^{\infty}\frac{(-1)^{ab+c+g}(\theta c)^{d}(\alpha d)^{g}}{bd!g!(e^{\theta}-1)^{ab}}E\left[X^{-\beta g}\right].$$

Also,

$$E\left[e^{-\alpha x^{-\beta}}\right] = \sum_{l=0}^{\infty}\frac{(-1)^{l}\alpha^{l}E\left[X^{-\beta l}\right]}{l!}.$$

$$
\begin{aligned}
E[\log(x)] &= E[\log(1+(x-1))] \\
&= \sum_{p=1}^{\infty} \frac{(-1)^{p+1}(x-1)^p}{p} \\
&= \sum_{p=1}^{\infty}\sum_{w=0}^{p} \binom{p}{w} \frac{(-1)^{p+1}(-1)^{p-w}E[X^p]}{p} \\
&= \sum_{p=1}^{\infty}\sum_{w=0}^{p} \binom{p}{w} \frac{(-1)^{2p+1-w}E[X^p]}{p} \\
&= -\sum_{p=1}^{\infty}\sum_{w=0}^{p} \binom{p}{w} \frac{(-1)^{-w}E[X^p]}{p}.
\end{aligned}
$$

Using results from the r^{th} moment in 2.10, the Shannon entropy of the KIWP distribution is given by

$$
\begin{aligned}
H[f(X)] &= \log\left[\frac{e^{\theta}-1}{ab\alpha\beta\theta}\right] - (\beta+1)\sum_{j,k,m=0}^{\infty}\sum_{p=1}^{\infty}\sum_{w=0}^{p}\binom{p}{w}\frac{(-1)^{-w}\omega_{j,k,m}(a,b,\theta)}{p} \\
&\times [\alpha(m+1)]^{\frac{p}{\beta}}\Gamma\left(1-\frac{p}{\beta}\right) + \sum_{j,k,m=0}^{\infty}\frac{\omega_{j,k,m}(a,b,\theta)}{m+1}
\end{aligned}
$$

$$
\begin{aligned}
&- \theta\sum_{j,k,m=0}^{\infty}\sum_{l=0}^{\infty}\frac{(-1)^l\alpha^l\omega_{j,k,m}(a,b,\theta)}{l![\alpha(m+1)]^l}\Gamma(1+l) \\
&+ (a-1)\Bigg\{\log(1-e^{\theta}) + \sum_{j,k,m=0}^{\infty}\sum_{q=1}^{\infty}\sum_{s=0}^{\infty}\sum_{u=0}^{\infty}\frac{\omega_{j,k,m}(a,b,\theta)(-1)^u\theta^s q^{s-1}(\alpha s)^u}{s!u!} \\
&\times \frac{\Gamma(1+u)}{[\alpha(m+1)]^u}\Bigg\} \\
&+ (b-1)\Bigg\{\sum_{j,k,m=0}^{\infty}\sum_{b=1}^{\infty}\sum_{c=0}^{\infty}\sum_{d=0}^{\infty}\sum_{g=0}^{\infty}\frac{\omega_{j,k,m}(a,b,\theta)(-1)^{ab+c+g}(\theta c)^d(\alpha d)^g}{bd!g!(e^{\theta}-1)^{ab}} \\
&\times \frac{\Gamma(1+g)}{[\alpha(m+1)]^g}\Bigg\}.
\end{aligned}
$$

Note that $\Gamma(a+1)=a!$

Reliability

Reliability may be defined as the probability that a piece of equipment operated under specific conditions performs to a specified standard for a particular period of time. It is a number between 0 and 1. It focuses on maximizing the inherent repeatability or consistency in an experiment. To maintain reliability internally, a researcher will use as many repeat sample groups as possible. It is a crucial part of any research design. If any test is not reliable, then its validity is questionable thus making the entire experiment a waste of time. Reliability R when X and Y have independent $KIWP(\alpha_1, \beta_1, \theta_1, a_1, b_1)$ and $KIWP(\alpha_2, \beta_2, \theta_2, a_2, b_2)$ is given by:

$$\begin{aligned} R = P(X > Y) &= \int_0^\infty f_x(x, \alpha_1, \beta_1, \theta_1, a_1, b_1) F_Y(x, \alpha_2, \beta_2, \theta_2, a_2, b_2) dx \\ &= \int_0^\infty a_1 b_1 \theta_1 \alpha_1, \beta_1 x^{-(\beta+1)} e^{-\alpha_1 x^{-\beta_1}} \exp(\theta_1 e^{-\alpha_1 x^{-\beta_1}}) \\ &\times \quad \frac{\exp(\theta_1 e^{-\alpha_1 x^{-\beta_1}} - 1)}{e^{\theta_1 - 1}}^{a_1 - 1} \times \left[1 - \frac{\exp(\theta_1 e^{-\alpha_1 x^{-\beta_1}} - 1)}{e^{\theta_1 - 1}}^{a_1}\right]^{(b_1 - 1)} \\ &\times \quad \left[1 - \left\{1 - \frac{1 - \exp(\theta_2 e^{-\alpha_2 x^{-\beta_2}})}{e^{\theta_2 - 1}}^{a_2}\right\}\right]^{b_2}. \end{aligned}$$

Using the binomial expansion $(1-z)^b = \sum_{i=0}^{\infty} \binom{b}{i} (-1)^i (z)^i$,

$$\left[1 - \frac{\exp(\theta_1 e^{-\alpha_1 x^{-\beta_1}} - 1)}{e^{\theta_1 - 1}}^{a_1}\right]^{(b_1 - 1)} = \sum_{i=0}^{\infty} (-1)^i \binom{b_1 - 1}{i} \frac{\exp(\theta_1 e^{-\alpha_1 x^{-\beta_1}} - 1)}{1 - e^{\theta_1}}^{a_1 i}.$$

Thus

$$\begin{aligned} \left[1 - \left\{1 - \frac{1 - \exp(\theta_2 e^{-\alpha_2 x^{-\beta_2}})}{e^{\theta_2 - 1}}^{a_2}\right\}\right]^{b_2} &= \sum_{j=0}^{\infty} \binom{b_2}{j} (-1)^j \\ &\times \quad \left\{1 - \frac{1 - \exp(\theta_2 e^{-\alpha_2 x^{-\beta_2}})}{1 - e^{\theta_2}}^{a_2}\right\}^j. \end{aligned}$$

$$\left\{1-\left(\frac{1-\exp(\theta_2 e^{-\alpha_2 x^{-\beta_2}})}{1-e^{\theta_2}}^{a_2}\right)\right\}^j = \sum_{k=0}^{\infty}\binom{j}{k}(-1)^k\left(\frac{1-\exp(\theta_2 e^{-\alpha_2 x^{-\beta_2}})}{1-e^{\theta_2}}\right)^{a_2}$$

Therefore,

$$\begin{aligned}\left[1-\left\{1-\frac{1-\exp(\theta_2 e^{-\alpha_2 x^{-\beta_2}})}{e^{\theta_2 - 1}}^{a_2}\right\}\right]^{b_2} &= \sum_{l=0}^{\infty}\sum_{k=0}^{l}\sum_{j=0}^{k}\binom{j}{k}\binom{b_2}{j}\binom{a_2 k}{l}\exp((-1)^{j+k+1}\\ &\times (1-e^{\theta_2})^{-a_2 k}\times\exp(\theta_2 e^{-\alpha_2 x^{-\beta_2}})^l.\end{aligned}$$

We further expand the expression by applying series expansions

$$\begin{aligned}\left(\frac{\exp(\theta_1 e^{-\alpha_1 x^{-\beta_1}}-1)}{1-e^{\theta_1}}\right)^{a_1 i} &= (-1)^{a_1 i+m}\sum_{m=0}^{\infty}\binom{a_1}{m}\left\{\exp(\theta_1 m e^{-\alpha_1 x^{-\beta_1}})\right\}^m(1-e_2^{\theta})^{-a_1 i}\\ &= (-1)^{a_1 i+m}\sum_{m=0}^{\infty}\binom{a_1 i}{m}\exp(\theta_1 m e^{-\alpha_1 x^{-\beta_1}})(1-e_1^{\theta})^{-a_1 i}.\end{aligned}$$

Expanding further by binomial expansions yields:

$$\left(1-\frac{1-\exp(\theta_1 e^{-\alpha_1 x^{-\beta_1}}}{e^{\theta_1 - 1}}\right)^{a_1-1} = (e_1^{\theta}-1)^{1-a_1}\sum_{n=0}^{\infty}\binom{a_1-1}{n}(-1)^n\exp(\theta_1 n e^{-\alpha_1 x^{-\beta_1}}).$$

Thus we have

$$\begin{aligned}R &= a_1 b_1 \theta_1 \alpha_1 \beta_1 \sum_{n=0}^{\infty}\sum_{m=0}^{n}\sum_{l=0}^{m}\sum_{k=0}^{l}\sum_{j=0}^{k}\sum_{i=0}^{j}\int_0^{\infty} x^{-(\beta+1)}\times(-1)^{a_1-1+i+n+ai+m+j+k+l}\\ &\times (e_1^{\theta}-1)^{1-a_1}(1-e_2^{\theta})^{-a_2 k - a_1 i}\binom{a_1-1}{n}\binom{b_1-1}{i}\binom{a_1 i}{m}\binom{j}{k}\binom{b_2}{j}\binom{a_2 k}{l}\\ &\times \exp(\theta_1 n e^{-\alpha_1 x^{-\beta_1}})\times\exp(\theta_1 m e^{-\alpha_1 x^{-\beta_1}})\times\exp(\theta_2 l e^{-\alpha_2 x^{-\beta_2}})dx.\end{aligned}$$

Notice that

$$e^t = \sum_{m=0}^{\infty}\frac{t^m}{m!}.$$

Thus, applying the above identity to $\exp(\theta_1 z e^{-\alpha_2 x^{-\beta_2}})$ yields

$$\exp(\theta_1 z e^{-\alpha_2 x^{-\beta_2}}) = \sum_{p=0}^{\infty} \frac{\theta_1^p z^p e^{-\alpha_1 p x^{-\beta_1}}}{p!}.$$

Likewise, we can apply the same identity to the expressions for $\exp(\theta_1 e^{-\alpha_1 x^{-\beta_1}})$, $\exp(\theta_1 m e^{-\alpha_1 x^{-\beta_1}})$ and $\exp(\theta_2 l e^{-\alpha_1 x^{-\beta_2}})$.

Thus, let $\int_0^\infty f_x(x, \alpha_1, \beta_1, \theta_1, a_1, b_1) F_Y(x, \alpha_2, \beta_2, \theta_2, a_2, b_2) dx$ be denoted $\int_0^\infty f_x(.) F_Y(.) dx$ Thus,

$$\begin{aligned}
\int_0^\infty f_x(.) F_Y(.) &= a_1 b_1 \theta_1 \alpha_1 \beta_1 \sum_{i,j,k,l,m,n=0}^{\infty} \sum_{p,q,r=0}^{\infty} \frac{\theta_1^{p+q+r} z^p m^r n^v}{p! q! r! v!} \\
&\times (-1)^{a_1(1+j)+i+j+k+m+n+l-1} \\
&\times \binom{a_1-1}{i} \binom{b_1-1}{j} \binom{b_2}{l} \binom{l}{m} \\
&\times \binom{am}{n} (e^{\theta_2} - 1)^{-am} \\
&\times (e^{\theta_1} - 1)^{1-a_1-a_1 j} \theta_2^v \\
&\times \int_0^\infty x^{-(\beta+1)} e^{-\alpha_1(1+p+q+r) x^{-\beta_1}} e^{-\alpha_2 v x^{-\beta_2}} dx.
\end{aligned}$$

By invoking the identity $e^t = \sum_{m=0}^{\infty} \frac{t^m}{m!}$, we can express $\exp(-\alpha_2 v x^{-\beta_2})$ as

$$\exp(-\alpha_2 v x^{-\beta_2}) = \sum_{t=0}^{\infty} \frac{(-\alpha_2)^t v^t x^{-t\beta_2}}{t!} = \sum_{t=0}^{\infty} \frac{(-1)^t (\alpha_2)^t v^t x^{-t\beta_2}}{t!}.$$

$$\begin{aligned}
R &= a_1 b_1 \theta_1 \alpha_1 \beta_1 \times \sum_{i,j,k,l,m,n=0}^{\infty} \sum_{p,q,r=0}^{\infty} \frac{\theta_1^{p+q+r} z^p m^r n^v}{p! q! r! v! t!} (-1)^{a_1(1+j)+i+j+k+m+n+l-1} \binom{a_1-1}{i} \\
&\times \binom{b_1-1}{j} \binom{b_2}{l} \binom{l}{m} \binom{am}{n} (e^{\theta_2} - 1)^{-am} (e^{\theta_1} - 1)^{1-a_1-a_1 j} \theta_2^v \\
&\times \int_0^\infty x^{-\beta_1 - t\beta_2 - 1} e^{-\alpha(1+p+q+r) x^{-\beta_1}} dx.
\end{aligned}$$

For the integration of the expression

$$\int_0^\infty x^{-\beta_1 - t\beta_2 - 1} e^{-\alpha_1(1+p+q+r)x^{-\beta_1}} dx,$$

we will let

$$u = \alpha_1(1+p+q+r)x^{-\beta_1},$$

and so

$$du = -\alpha_1\beta_1(1+p+q+r)x^{-\beta_1-1}dx.$$

Therefore

$$dx = \frac{-1}{\alpha_1\beta_1(1+p+q+r)} x^{\beta_1+1} du$$

$$= (1+p+q+r)^{\frac{1}{\beta_1}} \alpha_1^{\frac{1}{\beta_1}} \int_0^\infty u^{(\frac{t\beta_2}{\beta_1}+1)-1} e^{-u} du.$$

Therefore, we arrive at the expression below for the reliability:

$$\begin{aligned} R &= a_1 b_1 \theta_1 \alpha_1 \beta_1 \sum_{i,j,k,l,m,n=0}^{\infty} \sum_{p,q,r,t=0}^{\infty} \theta_2^v \theta_1^{p+q+r} \frac{\alpha^t v^t z^p m^r l^v}{v!q!p!r!t!} (-1)^{a_1(1+j)+i+j+k+m+n+l-1} \\ &\times \binom{a_1-1}{i}\binom{b_1-1}{j}\binom{b_2}{l}\binom{l}{m}\binom{am}{n}(e^{\theta_2}-1)^{-am}(e^{\theta_1}-1)^{1-a_1-a_1 j}(1+p+q+r)^{\frac{1}{\beta_1}} \\ &\times \alpha_1^{\frac{1}{\beta_1}} \Gamma\left(\frac{t\beta_2}{\beta_1}+1\right). \end{aligned}$$

Maximum Likelihood Estimation

In this section, the maximum likelihood method is used in estimating the parameters $a, b, \alpha, \beta, \theta$. The pdf of the KIWP distribution can be rewritten as

$$
\begin{aligned}
f(x) &= \frac{ab\theta\alpha\beta x^{-\beta-1} e^{-\alpha x^{-\beta}} \exp(\theta e^{-\alpha x^{-\beta}})}{e^{\theta}-1} \left[\frac{\exp(\theta e^{-\alpha x^{-\beta}})-1}{e^{\theta}-1}\right]^{a-1} \\
&\times \left[1-\left(\frac{\exp(\theta e^{-\alpha x^{-\beta}})-1}{e^{\theta}-1}\right)^{a}\right]^{b-1} \\
&= \left[\frac{ab\theta\alpha\beta}{(e^{\theta}-1)^{ab}}\right] x^{-\beta-1} e^{-\alpha x^{-\beta}} \exp(\theta e^{-\alpha x^{-\beta}}) \left[\exp(\theta e^{-\alpha x^{-\beta}})-1\right]^{a-1} \\
&\times \left[(e^{\theta}-1)^{a} - \left(\exp[\theta e^{-\alpha x^{-\beta}}]-1\right)^{a}\right]^{b-1} \\
&= \left[\frac{ab\theta\alpha\beta}{(e^{\theta}-1)^{ab}}\right] x^{-\beta-1} e^{-\alpha x^{-\beta}} \exp(\theta e^{-\alpha x^{-\beta}}) V^{a-1}(x) \left[(e^{\theta}-1)^{a} - V^{a}(x)\right],
\end{aligned}
$$

where

$$V(x) = \exp(\theta e^{-\alpha x^{-\beta}}) - 1.$$

Let $x_1, \cdots, x_n$ be a random sample from the KIWP distribution. The log-likelihood function is given by

$$
\begin{aligned}
L &= n\log(a) + n\log(b) + n\log(\alpha) + n\log(\beta) + n\log(\theta) - nab\log(e^{\theta}-1) \\
&- (\beta+1)\sum_{i=1}^{n}\log(x_i) - \alpha\sum_{i=1}^{n} x_i^{-\beta} + \theta\sum_{i=1}^{n} e^{-\alpha x_i^{-\beta}} + (a-1)\sum_{i=1}^{n}\log[V(x_i)] \\
&+ (b-1)\sum_{i=1}^{n}\log\left\{(e^{\theta}-1)^{a} - V^{a}(x_i)\right\}. \qquad (3.1)
\end{aligned}
$$

The associated score function is given by

$$U(\Theta) = (\frac{\partial L}{\partial a}, \frac{\partial L}{\partial b}, \frac{\partial L}{\partial \alpha}, \frac{\partial L}{\partial \beta}, \frac{\partial L}{\partial \theta}). \qquad (3.2)$$

The components of the score function are the partial derivatives with respect to each of the parameters. The maximum likelihood estimates of Θ can be obtained by solving the non-linear system of equations

$$U_n(\Theta) = 0.$$

Since the equations 3.1 and 3.2 are not in closed form, the solutions can be found by use of a numerical method such as the Newton Rhapson method. The elements of the score vector are given by;

$$\begin{aligned}\frac{\partial L}{\partial a} &= \frac{n}{a} - nb\log(e^{\theta}-1) + \sum_{i=1}^{n}\log[V(x_i)] \\ &+ (b-1)\sum_{i=1}^{n}\left\{\frac{(e^{\theta}-1)^a\log(e^{\theta}-1) - V^a(x)\log[V(x_i)]}{(e^{\theta}-1)^a - [V(x_i)]^a}\right\},\end{aligned}$$

$$\frac{\partial L}{\partial b} = \frac{n}{b} - na\log(e^{\theta}-1) + \sum_{i=1}^{n}\log\left\{(e^{\theta}-1)^a - V^a(x_i)\right\},$$

$$\begin{aligned}\frac{\partial L}{\partial \alpha} &= \frac{n}{\alpha} - \sum_{i=1}^{n}x_i^{-\beta} - \theta\sum_{i=1}^{n}x_i^{-\beta}\exp(-\alpha x_i^{-\beta}) + (a-1)\sum_{i=1}^{n}\frac{\frac{\partial V(x_i)}{\partial \alpha}}{V(x_i)} \\ &- (b-1)\sum_{i=1}^{n}\frac{a[V(x_i)]^{a-1}\frac{\partial V(x_i)}{\partial \alpha}}{(e^{\theta}-1)^a - [V(x_i)]^a},\end{aligned}$$

$$\begin{aligned}\frac{\partial L}{\partial \beta} &= \frac{n}{\beta} - \sum_{i=1}^{n}\log(x_i) + \alpha\sum_{i=1}^{n}x_i^{-\beta}\log(x_i) + \alpha\theta\sum_{i=1}^{n}x_i^{-\beta}\log(x_i)\exp(-\alpha x_i^{-\beta}) \\ &+ (a-1)\sum_{i=1}^{n}\frac{\frac{\partial V(x_i)}{\partial \beta}}{V(x_i)} - (b-1)\sum_{i=1}^{n}\frac{a[V(x_i)]^{a-1}\frac{\partial V(x_i)}{\partial \beta}}{(e^{\theta}-1)^a - [V(x_i)]^a}\end{aligned}$$

and

$$\begin{aligned}\frac{\partial L}{\partial \theta} &= \frac{n}{\theta} - \frac{nabe^{\theta}}{e^{\theta}-1} + \sum_{i=1}^{n} \exp(-\alpha x_i^{-\beta}) + (a-1)\sum_{i=1}^{n} \frac{\frac{\partial V(x_i)}{\partial \theta}}{V(x_i)} \\ &+ a(b-1)\sum_{i=1}^{n} \frac{e^{\theta}(e^{\theta}-1)^{a-1} - [V(x_i)]^{a-1}\frac{\partial V(x_i)}{\partial \theta}}{(e^{\theta}-1)^{a} - [V(x_i)]^{a}}.\end{aligned}$$

The mixed partial derivatives are given by:

$$\frac{\partial^2 L}{\partial a^2} = -\frac{n}{a^2} + \sum_{i=1}^{n} \frac{\partial V(x_i)}{\partial \theta} V(x_i),$$

$$\begin{aligned}\frac{\partial^2 L}{\partial a \partial b} &= -n\log(e^{\theta}-1) + \sum_{i=1}^{n} \log[V(x_i)] \\ &+ \sum_{i=1}^{n} \left\{ \frac{(e^{\theta}-1)^{a}\log(e^{\theta}-1) - V^{a}(x)\log[V(x_i)]}{(e^{\theta}-1)^{a} - [V(x_i)]^{a}} \right\},\end{aligned}$$

$$\begin{aligned}\frac{\partial^2 L}{\partial a \partial \alpha} &= -(b-1)\sum_{i=1}^{n} \left\{ \left[(e^{\theta}-1)^{a} - [V(x_i)]^{a} \right] \left[a[V(x_i)]^{a-1}\log[V(x_i)] \frac{\partial V(x_i)}{\partial \alpha} \right.\right. \\ &+ \left. [V(x_i)]^{a-1}\frac{\partial V(x_i)}{\partial \alpha} \right] - (b-1)\left[(e^{\theta}-1)^{a}\log(e^{\theta}-1) - V^{a}(x)\log[V(x_i)] \right] \\ &\times \left.\left[a[V(x_i)]^{a-1}\frac{\partial V(x_i)}{\partial \alpha} \right] \right\} \left[(e^{\theta}-1)^{a} - [V(x_i)] \right]^{-2} + \sum_{i=1}^{n} \frac{\frac{\partial V(x_i)}{\partial \alpha}}{V(x_i)},\end{aligned}$$

$$\begin{aligned}\frac{\partial^2 L}{\partial a \partial \beta} &= -(b-1)\sum_{i=1}^{n} \left\{ \left[(e^{\theta}-1)^{a} - [V(x_i)]^{a} \right] \left[a[V(x_i)]^{a-1}\log[V(x_i)] \frac{\partial V(x_i)}{\partial \beta} \right.\right. \\ &+ \left. [V(x_i)]^{a-1}\frac{\partial V(x_i)}{\partial \beta} \right] - (b-1)\left[(e^{\theta}-1)^{a}\log(e^{\theta}-1) - V^{a}(x)\log[V(x_i)] \right] \\ &\times \left.\left[a[V(x_i)]^{a-1}\frac{\partial V(x_i)}{\partial \beta} \right] \right\} \left[(e^{\theta}-1)^{a} - [V(x_i)] \right]^{-2} + \sum_{i=1}^{n} \frac{\frac{\partial V(x_i)}{\partial \beta}}{V(x_i)},\end{aligned}$$

$$
\begin{aligned}
\frac{\partial^2 L}{\partial a \partial \theta} &= \frac{\frac{\frac{\partial V(x_i)}{\partial \theta}}{V(x_i)} - nb\frac{e^\theta}{e^\theta - 1} + (b-1)\sum_{i=1}^n [(e^\theta - 1)^a - V^a(x_i)]}{[(e^\theta - 1)^a - [V^a(x_i)]^2} \times A \\
&- \frac{(e^\theta - 1)\log(e^\theta - 1) - V^a(x_i)\log[V(x_i)] \times a(e^\theta - 1)^{a-1}e^\theta - aV^{a-1}(x_i)\frac{\partial V(x_i)}{\partial \theta}}{[(e^\theta - 1)^a - [V^a(x_i)]^2},
\end{aligned}
$$

where A is denoted by

$$
\begin{aligned}
A &= a(e^\theta - 1)\log(e^\theta - 1) + (e^\theta - 1)^a + (e^\theta - 1)\frac{e^\theta}{e^\theta - 1} - aV^{a-1}(x_i)\frac{\partial V(x_i)}{\partial \theta}\log[V(x_i)]. \\
&- V^a(x_i)\frac{\frac{\partial V(x_i)}{\partial \beta}}{V(x_i)}
\end{aligned}
$$

$$
\frac{\partial^2 L}{\partial b^2} = \frac{-n}{b^2}
$$

,

$$
\frac{\partial^2 L}{\partial b \partial \alpha} = -\sum_{i=1}^n \frac{a[V(x_i)]^{a-1}\frac{\partial V(x_i)}{\partial \alpha}}{(e^\theta - 1)^a - [V(x_i)]^a},
$$

$$
\frac{\partial^2 L}{\partial b \partial \beta} = -\sum_{i=1}^n \frac{a[V(x_i)]^{a-1}\frac{\partial V(x_i)}{\partial \beta}}{(e^\theta - 1)^a - [V(x_i)]^a},
$$

$$
\frac{\partial^2 L}{\partial b \partial \theta} = \frac{-nae^\theta}{e^\theta - 1} + a\sum_{i=1}^n \frac{(e^\theta - 1)^{a-1}e^\theta - [V(x_i)]^{a-1}\frac{\partial V(x_i)}{\partial \theta}}{(e^\theta - 1)^a - [V(x_i)]^a},
$$

$$\begin{aligned}
\frac{\partial^2 L}{\partial \alpha^2} &= \frac{-n}{\alpha^2} - \theta \sum_{i=1}^{n} x_i^{-2\beta} \exp(-\alpha x_i^{-\beta}) + (a-1) \sum_{i=1}^{n} \frac{V(x_i)\frac{\partial^2 V(x_i)}{\partial \alpha^2} - \left(\frac{\partial V(x_i)}{\partial \alpha}\right)^2}{[V(x_i)]^2} \\
&- a(b-1) \sum_{i=1}^{n} \Bigg\{ \Bigg[\Bigg((e^\theta - 1)^a - [V(x_i)]^a \Bigg) \Bigg\{ (a-1)[V(x_i)]^{a-2} \left(\frac{\partial V(x_i)}{\partial \alpha}\right)^2 \\
&+ [V(x_i)]^{a-1} \frac{\partial^2 V(x_i)}{\partial \alpha^2} \Bigg\} + a^2 [V(x_i)]^{2a-2} \left(\frac{\partial V(x_i)}{\partial \alpha}\right)^2 \Bigg] \\
&\times \left[(e^\theta - 1)^a - [V(x_i)]^a) \right]^{-2} \Bigg\},
\end{aligned}$$

$$\begin{aligned}
\frac{\partial^2 L}{\partial \alpha^2} &= \frac{-n}{\alpha^2} - \theta \sum_{i=1}^{n} x_i^{-2\beta} \exp(-\alpha x_i^{-\beta}) + (a-1) \sum_{i=1}^{n} \frac{V(x_i)\frac{\partial^2 V(x_i)}{\partial \alpha^2} - \left(\frac{\partial V(x_i)}{\partial \alpha}\right)^2}{[V(x_i)]^2} \\
&- a(b-1) \sum_{i=1}^{n} \Bigg\{ \Bigg[\Bigg[V(x_i)]^{a-1} ((e^\theta - 1)^a - [V(x_i)]^a \Bigg) \Bigg\{ (a-1)[V(x_i)]^{-1} \left(\frac{\partial V(x_i)}{\partial \alpha}\right)^2 \\
&+ \frac{\partial^2 V(x_i)}{\partial \alpha^2} \Bigg\} + a^2 [V(x_i)]^{-1} \left(\frac{\partial V(x_i)}{\partial \alpha}\right)^2 \Bigg] \\
&\times \left[(e^\theta - 1)^a - [V(x_i)]^a) \right]^{-2} \Bigg\},
\end{aligned}$$

$$\begin{aligned}
\frac{\partial^2 L}{\partial \alpha \partial \beta} &= \sum_{i=1}^{n} [\log(x_i)] \sum_{i=1}^{n} x_i^{-\beta} + (a-1) \sum_{i=1}^{n} \frac{\frac{\partial^2 V(x_i)}{\partial \alpha \partial \beta} V(x_i) - \frac{\partial V(x_i)}{\partial \alpha} \frac{\partial V(x_i)}{\partial \beta}}{[V(x_i)]^2]} \\
&- \sum_{i=1}^{n} (a-1) \frac{V^{a-2}(x) \frac{\partial V(x_i)}{\partial \beta} \frac{\partial V(x_i)}{\partial \alpha} + V^{a-1}(x) \frac{\partial^2 V(x_i)}{\partial \alpha \partial \beta}}{[(e^\theta - 1)^a - [V(x_i)]]} \\
&- a(b-1) \sum_{i=1}^{n} \frac{\frac{V^{a-1}(x) \partial V(x_i)}{\partial \alpha} [a(e^\theta - 1)^{a-1} e^\theta - a V^{a-1}(x_i) \frac{\partial V(x_i)}{\partial \theta}]}{[(e^\theta - 1)^a - [V(x_i)]]^2},
\end{aligned}$$

$$\frac{\partial^2 L}{\partial\alpha\partial\theta} = -\sum_{i=1}^{n} x_i^{-\beta}\exp(-\alpha x_i^{-\beta}) + (a-1)\sum_{i=1}^{n}\frac{V(x_i)\frac{\partial^2 V(x_i)}{\partial\alpha\partial\theta} - \frac{\partial V(x_i)}{\partial\alpha}\frac{\partial V(x_i)}{\partial\theta}}{[V(x_i)]^2},$$

$$\begin{aligned}\frac{\partial^2 L}{\partial\beta^2} &= \frac{-n}{\beta^2} - \alpha\sum_{i=1}^{n} x_i^{-\beta}[\log(x_i)]^2 + \alpha\theta\sum_{i=1}^{n} x_i^{-\beta}[\log(x_i)]^2\exp(-\alpha x_i^{-\beta})(\alpha-1)\\ &+ a-1)\sum_{i=1}^{n}\frac{V(x_i)\frac{\partial^2 V(x_i)}{\partial\beta^2} - \left(\frac{\partial L}{\partial\beta}\right)}{[V(x)]^2},\end{aligned}$$

$$\begin{aligned}\frac{\partial^2 L}{\partial\beta\partial\theta} &= \alpha\sum_{i=1}^{n} x_i^{-\beta}\exp(-\alpha x_i^{-\beta})\log(x_i) + (a-1)\sum_{i=1}^{n}\frac{V(x_i)\frac{\partial^2 V(x_i)}{\partial\beta\partial\theta} - \frac{\partial V}{\partial\beta}\frac{\partial V}{\partial\theta}}{[V(x)]^2}\\ &- a(b-1)\sum_{i=1}^{n}\frac{(a-1)V^{a-2}(x_i)\frac{\partial V}{\partial\theta}\frac{\partial V}{\partial\beta} + [V(x_i)]^{a-1}\frac{\partial^2 V}{\partial\beta\partial\theta}[(e^\theta-1)^a - V(x_i)^a]}{[(e^\theta-1)^a - [V(x_i)]^a]}\\ &- \frac{a(e^\theta-1)^{a-1}e^\theta - aV^{a-1}(x_i)\frac{\partial V}{\partial\theta})\left[aV^{a-1}(x_i)\frac{\partial V}{\partial\beta}\right]}{[(e^\theta-1)^a - [V(x_i)]^a]}\end{aligned}$$

and

$$\begin{aligned}\frac{\partial^2 L}{\partial\theta^2} &= \frac{-n}{\alpha^2} + nab\frac{e^\theta}{(e^\theta-1)^2} + (a-1)\sum_{i=1}^{n}\frac{\partial^2 V(x_i)\partial\theta^2 V(x_i) - \frac{\partial V}{\partial\theta}\frac{\partial V}{\partial\theta}}{[V(x)]^2}\\ &+ \frac{a(a-1)(e^\theta-1)^{a-2}e^{2\theta} + a(e^\theta-1)e^\theta - a(a-1)V^{a-2}(x_i)\left(\frac{\partial V}{\partial\theta}\right)^2 aV^{a-1}(x_i)\frac{\partial^2 V}{\partial\theta^2}}{[(e^\theta-1)^a - [V(x_i)]^a]^2}.\end{aligned}$$

For the one variable partials, we obtain

$$\begin{aligned}
\frac{\partial V(x_i)}{\partial a} &= 0, \\
\frac{\partial V(x_i)}{\partial b} &= 0, \\
\frac{\partial V(x_i)}{\partial \alpha} &= -\exp[\theta e^{-\alpha x^{-\beta}}]\theta x^{-\beta}\exp[-\alpha x^{-\beta}], \\
\frac{\partial V(x_i)}{\partial \theta} &= e^{-\alpha x^{-\beta}}\exp[\theta e^{-\alpha x^{-\beta}}], \\
\frac{\partial V(x_i)}{\partial \beta} &= \theta\alpha\beta x^{-\beta-1}e^{-\alpha x^{-\beta}}\exp[\theta e^{-\alpha x^{-\beta}}]
\end{aligned}$$

Fisher Information Matrix

In this section, we present a measure for the amount of information. This information can be used for interval estimation and hypothesis testing for the model parameters $a, b, \alpha, \beta, \theta$. Let X be a random variable with the KIWP density $f_{KIWP}(.;\Theta)$, where

$$\begin{aligned}
\Theta &= (\theta_1, \theta_2, \theta_3, \theta_4, \theta_5)^T \\
&= (a, b, \alpha, \beta, \theta)^T.
\end{aligned}$$

The Fisher information matrix (FIM) is the 5×5 symmetric matrix denoted as follows:

$$\begin{bmatrix}
\frac{\partial^2 L}{\partial a^2} & \frac{\partial^2 L}{\partial a \partial b} & \frac{\partial^2 L}{\partial a \partial \alpha} & \frac{\partial^2 L}{\partial a \partial \beta} & \frac{\partial^2 L}{\partial a \partial \theta} \\
\frac{\partial^2 L}{\partial b \partial a} & \frac{\partial^2 L}{\partial b^2} & \frac{\partial^2 L}{\partial b \partial \alpha} & \frac{\partial^2 L}{\partial b \partial \beta} & \frac{\partial^2 L}{\partial b \partial \theta} \\
\frac{\partial^2 L}{\partial \alpha \partial a} & \frac{\partial^2 L}{\partial \alpha \partial b} & \frac{\partial^2 L}{\partial \alpha^2} & \frac{\partial^2 L}{\partial \alpha \partial \beta} & \frac{\partial^2 L}{\partial \alpha \partial \theta} \\
\frac{\partial^2 L}{\partial \beta \partial a} & \frac{\partial^2 L}{\partial \beta \partial b} & \frac{\partial^2 L}{\partial \beta \alpha} & \frac{\partial^2 L}{\partial \beta^2} & \frac{\partial^2 L}{\partial \beta \partial \theta} \\
\frac{\partial^2 L}{\partial \theta \partial a} & \frac{\partial^2 L}{\partial \theta \partial b} & \frac{\partial^2 L}{\partial \theta \partial \alpha} & \frac{\partial^2 L}{\partial \theta \partial \beta} & \frac{\partial^2 L}{\partial \theta^2}
\end{bmatrix}$$

The elements

$$I_{ij}(\Theta) = -E_\theta \left[\frac{\partial^2 \log(f_{KIWP}(x, \Theta))}{\partial\theta_i \partial\theta_j} \right]$$

of the Fisher information matrix (FIM) can be obtained by considering the second order partial derivatives of the log likelihood function in equation 3.1. The elements can be obtained numerically by utilizing MATLAB software. The total FIM $I_n(\Theta)$ can be approximated by:

$$J_n(\hat{\Theta}) \approx \left[\frac{-\partial^2 \log L}{\partial\theta_i \partial_j} |_{\theta=\hat{\theta}} \right]_{5\times5}.$$

When dealing with actual real world data, the matrix above is obtained after convergence of the Newton Rhapson method in MATLAB.

Asymptotic Confidence Intervals

In this section, we present the asymptotic confidence intervals for the parameters of the KIWP distribution. The expectations in the Fisher's information Matrix (FIM) can be obtained numerically. Let $\hat{\Theta} = (\hat{a}, \hat{b}, \hat{\alpha}, \hat{\beta}, \hat{\theta})$ be the maximum likelihood estimate of $\Theta = (a, b, \alpha, \beta, \theta)$. Under the usual regularity conditions and that the parameters are in the interior of the parameter space, but not on the boundary, we thus have

$$\sqrt{n}(\hat{\Theta} - \Theta) \xrightarrow{d} N_5(\underline{0}, \boldsymbol{I}^{-1}(\Theta))$$

where $\boldsymbol{I}(\Theta)$ is the expected Fisher information matrix. The asymptotic behavior is still valid if $\boldsymbol{I}(\Theta)$ is replaced by the observed information matrix evaluated at $\hat{\Theta}$, that is $J(\hat{\Theta})$. The multivariate normal distribution with mean vector

$$(\underline{0}) = (0, 0, 0, 0, 0)^T$$

and covariance matrix $\boldsymbol{I}^{-1}(\Theta)$ can be used to construct confidence intervals for the model parameters. The approximate $100(1-\eta)\%$ two-sided confidence intervals for a,b, α,β,θ are given by:

$$\hat{a} \pm Z_{\frac{\eta}{2}}\sqrt{\boldsymbol{I}_{\boldsymbol{aa}}^{-1}(\Theta)}, \qquad \hat{b} \pm Z_{\frac{\eta}{2}}\sqrt{\boldsymbol{I}_{\boldsymbol{bb}}^{-1}(\Theta)},$$

$$\hat{\alpha} \pm Z_{\frac{\eta}{2}}\sqrt{\boldsymbol{I}_{\boldsymbol{\alpha\alpha}}^{-1}(\Theta)}, \qquad \hat{\beta} \pm Z_{\frac{\eta}{2}}\sqrt{\boldsymbol{I}_{\boldsymbol{\beta\beta}}^{-1}(\Theta)},$$

$$\hat{\theta} \pm Z_{\frac{\eta}{2}}\sqrt{\boldsymbol{I}_{\boldsymbol{\theta\theta}}^{-1}(\Theta)}$$

respectively where $\boldsymbol{I}_{\boldsymbol{aa}}^{-1}(\Theta)$, $\boldsymbol{I}_{\boldsymbol{bb}}^{-1}(\Theta)$, $\boldsymbol{I}_{\boldsymbol{\alpha\alpha}}^{-1}(\Theta)$, $\boldsymbol{I}_{\boldsymbol{\beta\beta}}^{-1}(\Theta)$ and $\boldsymbol{I}_{\boldsymbol{\theta\theta}}^{-1}(\Theta)$ are diagonal elements of

$$\boldsymbol{I}_{\boldsymbol{n}}^{-1}(\Theta) = (\mathrm{n\ I}\ (\hat{\Theta}))^{-1}$$

and $Z_{\frac{\eta}{2}}$ is the upper $(\eta/2)^{th}$ percentile of a standard normal distribution.

Concluding Remarks

Measures of uncertainty are discussed including Rényi Entropy, Shannon Entropy and reliability. We also discuss maximum likelihood estimation and we construct the Fisher information matrix and its various components.

CHAPTER 4

MONTE CARLO SIMULATION STUDY

Monte Carlo Simulation Study

In this section, we study the performance of the KIWP distribution by conducting various simulations for different example sizes and different parameter values. The algorithm I in the appendix is used to generate random data from the KIWP distribution. The simulation study is repeated for $N = 5,000$ times each with sample size $n = 25, 50, 75, 100, 200, 400$ and the parameter values I: $\alpha = 3.5, \beta = 3.0, \theta = 3.8, a = 0.5, b = 0.5$ and II: $\alpha = 3.3, \beta = 1.6, \theta = 3.3, a = 0.5, b = 0.6$. Three quantities are computed in this simulation study

(a) Average bias of the MLE $\hat{\theta}$ of the parameter $\theta = \alpha, \beta, \theta, a, b$ defined as

Average Bias $= \frac{1}{N}\sum_{i=1}^{N}(\hat{\theta} - \theta)$

(b) Root mean square error (RMSE) of the MLE $\hat{\theta}$ of the parameter $\theta = \alpha, \beta, \theta, a, b$:

RMSE $= \sqrt{\frac{1}{N}\sum_{i=1}^{N}(\hat{\theta} - \theta)^2}$

(c) The Average Width (AW) of 95% confidence intervals of the parameter.

Table 3 presents the Average bias, RMSE and AW values of the parameters $\alpha, \beta, \theta, a, b$ for different sample sizes. From the results, we can verify that as the sample size n increases, the RMSEs decay towards 0. We also observe that for all parametric values, the biases decrease as sample size n increases. Also, the table shows that the average confidence widths decrease as the sample size increases. Consequently, the MLE's and their asymptotic results can be used for estimating and constructing confidence intervals even for reasonably small sample sizes.

Table 3. *Monte Carlo Simulation Results: Average Bias, RMSE and AW*

		I			II		
Parameter	n	Average Bias	RMSE	AW	Average Bias	RMSE	AW
α	25	4.888	6.504	55.378	2.864	4.630	30.087
	50	3.681	5.497	38.599	3.979	5.790	44.635
	75	2.999	4.900	30.469	2.429	4.174	24.312
	100	2.639	4.446	25.695	1.977	3.694	19.766
	200	1.795	3.429	17.022	1.367	2.813	13.546
	400	0.956	2.348	10.742	0.922	2.092	9.177
β	25	0.0452	0.998	9.167	0.261	0.654	4.314
	50	0.219	0.993	7.575	0.215	0.698	5.381
	75	0.232	0.927	6.585	0.246	0.595	3.687
	100	0.256	0.872	5.919	0.250	0.571	3.332
	200	0.195	0.751	4.377	0.164	0.449	2.449
	400	0.102	0.604	3.169	0.099	0.355	1.754
θ	25	3.363	4.848	43.739	3.339	4.775	32.600
	50	2.776	4.535	34.679	4.229	5.519	43.662
	75	2.372	4.184	28.932	2.773	4.244	26.164
	100	2.119	3.925	25.071	2.544	3.986	22.850
	200	1.368	3.077	16.919	1.514	2.815	14.535
	400	0.780	2.233	11.204	0.825	1.925	9.331
a	25	-0.222	0.266	2.236	-0.188	0.272	2.239
	50	-0.173	0.286	2.616	-0.230	0.275	2.141
	75	-0.133	0.298	2.743	-0.159	0.270	2.192
	100	-0.123	0.287	2.438	-0.141	0.269	2.125
	200	-0.066	0.297	2.201	-0.081	0.278	1.972
	400	-0.005	0.312	1.928	-0.040	0.270	1.594
b	25	0.534	1.189	6.922	0.210	0.786	3.593
	50	0.220	0.695	3.133	0.475	1.154	7.172
	75	0.138	0.521	2.205	0.116	0.580	2.486
	100	0.082	0.395	1.658	0.060	0.465	1.966
	200	0.036	0.244	1.074	0.024	0.296	1.318
	400	0.020	0.156	0.756	0.014	0.209	0.930

Concluding Remarks

In this chapter, we carried out a simulation study to ascertain the precision of the maximum likelihood estimates of the KIWP model parameter estimates. We also discussed the random data generating algorithm processes.

CHAPTER 5

APPLICATIONS TO LIFETIME DATA

In this section, we present examples that illustrate the flexibility and the applicability of the KIWP distribution in modelling real world data. We fit the density functions of the KIWP distribution and the IWP. We also compare the KIWP to other comparable distributions such as the inverse Weibull distribution(IW) by Khan et al. (2008), the exponentiated Burr XII Poisson distribution (EBXIIP) by Ramos et al. (2015), the exponentiated Kum-Dagum distribution (EKD) by Huang and Oluyede (2014) and the gamma inverse Weibull distribution (GIW) by Pararai et al. (2014). The density functions for the EBXIIP, EKD and GIW distributions are respectively given by:

$$\begin{aligned} f_{EBXIIP}(c,k,s,a,\alpha) &= \exp\left(-\theta\left(1-\left\{1+\left(\frac{x}{s}\right)^c\right\}^{-k}\right)^{\alpha}\right)\times s^{-c}(1-\exp(-\theta))^{-1} \\ &\times \left(1+\left(\frac{x}{c}\right)^c\right)\times(ck\alpha\theta x^{c-1}) \\ &\times \left(1-\left\{1+\left(\frac{x}{s}\right)^c\right\}^{-k}\right)^{-\alpha}, c,k>0, s>0, a>0, \alpha>0, \end{aligned}$$

$$\begin{aligned} f_{EKD}(\alpha,\beta,\theta,a,b) &= \alpha\beta\theta ab x^{-\theta-1}(1+\beta x^{-\theta})^{-\alpha-1}(1-(1+\beta x^{-\theta})^{-\alpha})^{a-1} \\ &\times [(1-(1+\beta x^{-\theta})^{-\alpha})^a]^{b-1}, \alpha>0, \beta>0, a>0, b>0, \end{aligned}$$

and

$$f_{GIW}(x,\alpha,\beta,\theta)=\frac{\beta x^{-1}}{\Gamma(\theta)}ab^{\theta}\exp(-\alpha x^{-\beta}).\alpha>0,\beta>0,\theta>0$$

For each data set, the estimates of the parameters of the distributions and information criterion statistics are calculated. Estimates of the parameters of the distributions, Akaike information criterion ($AIC = 2p - 2\log(L)$), consistent Akaike information criterion ($AICC = AIC + \frac{2p(p+1)}{n-p-1}$), Bayesian information criterion ($BIC = p\log(n) - 2\log(L)$), where $L = L(\hat{\Delta})$ is the value of the likelihood function evaluated at the parameter estimates. The Akaike information criterion proposed by Akaike (1974) is defined by $AIC = -2L(\hat{\phi}; x_i) + 2k$ where $x_1, x_2, ..., x_n$ is the given random sample,$\hat{\phi}$ is the maximum likelihood estimator of ϕ and k is the length of the vector ϕ. The smaller the AIC, the better the model. When the sample size n is small or if the number of parameters k is large, the probability to select the model with many parameters will be increased using AIC. Hurvich and Tsai (1989) proposed a corrected Akaike information criterion AICc. $AICc = AIC + \frac{2k(k+1)}{n-k-1}$ where k is the length of the vector and n is the sample size. When sample size n is much larger than k, AICc converges to AIC. For small sample size, it is recommended to use AICc rather than AIC or in the case of many parameters. Bozdogan (1987) proposed consistent akaike information criterion (CAIC) which is defined by: $CAIC = -2L + k(\log(n) + 1)$. Bayesian information criteria due to Schwarz et al. (1978) is defined as follows: $BIC = k\log(n) - 2L(\hat{\phi}; x_i)$ where k is the length of the vector and n is sample size.

The Cramer Von Mises (W*), Anderson Darling (A*) and sum of squares from the probability plots are also presented. The Cramer Von Mises criterion is a criterion used for assessing the goodness of fit of a cdf F^* compared to a given empirical distribution function F_n or for comparing two empirical distributions. The Anderson Darling test is a statistical test of whether a given sample of data is drawn from a given probability distribution. The test assumes that there are no parameters to be estimated in the distribution being tested. When comparing models, the model with the smallest AIC is considered to be the best fit model for a given data set. However, when n is small or the number of parameters is large, the chance of selecting a model with many parameters as the best model will be increased using AIC. In such situations, the AICC is more suitable for best model selection. When n is large, the AICC converges to AIC. When selecting the best model with the smallest sum of squares (SS) is considered the best fit model.

Plots of the fitted densities, the histogram of the data and probability plots by Chambers et al. (1983) are given for each dataset. For the probability plot, we make a plot of $F_{KIWP}(x, \alpha, \beta, \theta, a, b)$ against $\frac{j-0.375}{n+0.25}$, $j = 1, 2, ..., n$ where $x_{(j)}$ are the ordered values of the observed data. The measures of closeness are given by the sum of squares.

$$SS = \sum_{j=1}^{n} \left[F_{KIWP}(x_{(j)} - \left(\frac{j - 0.375}{n + 0.25} \right)) \right]^2$$

The goodness of fit statistics W^* (Cramer Von Mises) and A^* (Anderson Darling) introduced by Chen and Balakrishnan (1995) were tabulated and used as an analysis tool for establishing the best fitting distribution for the particular dataset in question. As a rule of thumb, smaller values of A^* and W^* give a better distributional fit.

If $G(x, \Theta)$ is a cumulative distribution function where the form of G is known but the k-dimensional parameter vector Θ is unknown. We can obtain the statistics W^* and A^* by the following computation.

1. Calculate $U_i = G(x_i, \hat{\theta})$, where the x_i's are in ascending order.

2. Perform the computation of $t_i = \Phi^{-1}(u_i)$ where $\Phi(.)$ is the standard normal cdf and $\Phi^{-1}(.)$ the inverse.

3. Calculate $v_i = \Phi((t_i - \bar{t})/s_t)$ where $\bar{t} = n^{-1} \sum_{i=1}^{n} t_i$ and $s_t^2 = (n-1)^{-1} \sum_{i=1}^{n} (t_i - \bar{t})^2$.

4. Calculate $W^2 = \sum_{i=1}^{n} v_i - (2_i - 1)/(2n)^2 + 1/(12n)$ and
$A^2 = -n - n^{-1} \sum_{i=1}^{n} (2i-1)log(v_i) + (2n + 1 - 2i)log(1 - v_i)$.

5. Transform W^2 to $W^* = W^2 \left(1 + \frac{0.5}{n}\right)$.
Likewise, transforming A^2 to $A^* = A^2 \left(1 + \frac{0.75}{n} + \frac{2.25}{n^2}\right)$.

We can use the LR test to compare the fit of the KIWP distribution with its submodels for a given dataset. For example, to test $a = b = 1$, the LR statistic is $\omega = 2[\log(L(\hat{\alpha}, \hat{\beta}, \hat{\theta}, \hat{a}, \hat{b}) - \log(L(\tilde{\alpha}, \tilde{\beta}, \tilde{\theta}, 1, 1)]$ where $\hat{\alpha}, \hat{\beta}, \hat{\theta}, \hat{a}, \hat{b}$ are the unrestricted estimates and $\tilde{\alpha}, \tilde{\beta}, \tilde{\theta}$ are restricted estimates. The LR test rejects the null hypothesis if $\omega > \chi^2_d$ where χ^2_d denotes the upper $100d\%$ point of the χ^2 distribution with 2 degrees of freedom. We consider 4 datasets.

Cancer Patients Data Set

The first data set consists of data of cancer patients. The data represents the remission times (in months) of a random sample of 128 bladder cancer patients from Lee and Wang (2003). The starting point of the iterative processes for the cancer patients data set is $(1.0, 0.009, 10.0, 0.1, 0.1)$. Estimates of the parameters of the KIWP distribution, Akaike information criterion (AIC), consistent Akaike information criterion (AICC), Bayesian information criterion (BIC) are given in table 5 for cancer patients data. The sum of squares (SS) and goodness of fit statistics W* and A* are also given.

Table 4. *Remission Times (in months) of Bladder Cancer Patients.*

0.08	2.09	3.48	4.87	6.94	8.66	13.11	23.63	0.20	2.23
3.52	4.98	6.97	9.02	13.29	0.40	2.26	3.57	5.06	7.09
9.22	13.80	25.74	0.50	2.46	3.64	5.09	7.26	9.47	14.24
25.82	0.51	2.54	3.70	5.17	7.28	9.74	14.76	26.31	0.81
2.62	3.82	5.32	7.32	10.06	14.77	32.15	2.64	3.88	5.32
7.39	10.34	14.83	34.26	0.90	2.69	4.18	5.34	7.59	10.66
15.96	36.66	1.05	2.69	4.23	5.41	7.62	10.75	16.62	43.01
1.19	2.75	4.26	5.41	7.63	17.12	46.12	1.26	2.83	4.33
5.49	7.66	11.25	17.14	79.05	1.35	2.87	5.62	7.87	11.64
17.36	1.40	3.02	4.34	5.71	7.93	11.79	18.10	1.46	4.40
5.85	8.26	11.98	19.13	1.7	3.25	4.50	6.25	8.37	12.02
2.02	3.31	4.51	6.54	8.53	12.03	20.28	2.02	3.36	6.76
12.07	21.73	2.07	3.36	6.93	8.65	12.63	22.69		

Plots of the fitted densities and histogram, observed probability versus predicted probability for cancer patients data are given in figures 4 and 5.

Table 5. *Estimates Of Models For Cancer Patients Data*

	Estimates					Statistics						
Model	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\theta}$	$\hat{a}$	$\hat{b}$	$-2\log L$	AIC	$AICC$	BIC	SS	A^*	W^*
KIWP$(\alpha,\beta,\theta,a,b)$	1.901	0.735	20.527	0.257	5.041	817.9	827.9	828.4	842.1	0.011	0.074	0.012
	(0.713)	(0.222)	(14.037)	(0.219)	(4.008)							
IWP(α,β,θ)	0.636	1.002	6.481			854.3	860.3	860.5	868.8	0.354	2.532	0.402
	(0.146)	(0.058)	(1.325)									
IW(α,β,θ,a)	2.431	0.752				888.0	892.0	892.1	897.7	0.967	4.547	0.744
	(0.219)	(0.042)										
EKum-Dagum$(\alpha,\beta,\theta,a,b)$	2.243	30.286	0.401	1.539	1.842	819.3	829.3	829.8	843.6	0.015	0.114	0.017
	(4.382)	(82.654)	(0.882)	(1.519)	(1.298)							
GIW(α,β,θ)	81.374	0.111	67.506			832.7	840.65	840.83	849.20	0.168	1.120	0.168
	(54.820)	(0.040)	(49.819)									
	$\hat{c}$	$\hat{k}$	$\hat{s}$	$\hat{\alpha}$	$\hat{\beta}$							
EBXIIP(c,k,s,α,β)	1.386	0.214	16.633	1.007	14.542	819.6	829.6	830.1	843.9	0.018	0.133	0.020
	(1.240)	(1.006)	(18.245)	(1.002)	(31.918)							

Note. Standard errors are in parentheses

Plots of the fitted densities and histogram, observed probability versus predicted probability for cancer patients data are given in figures 4 and 5.

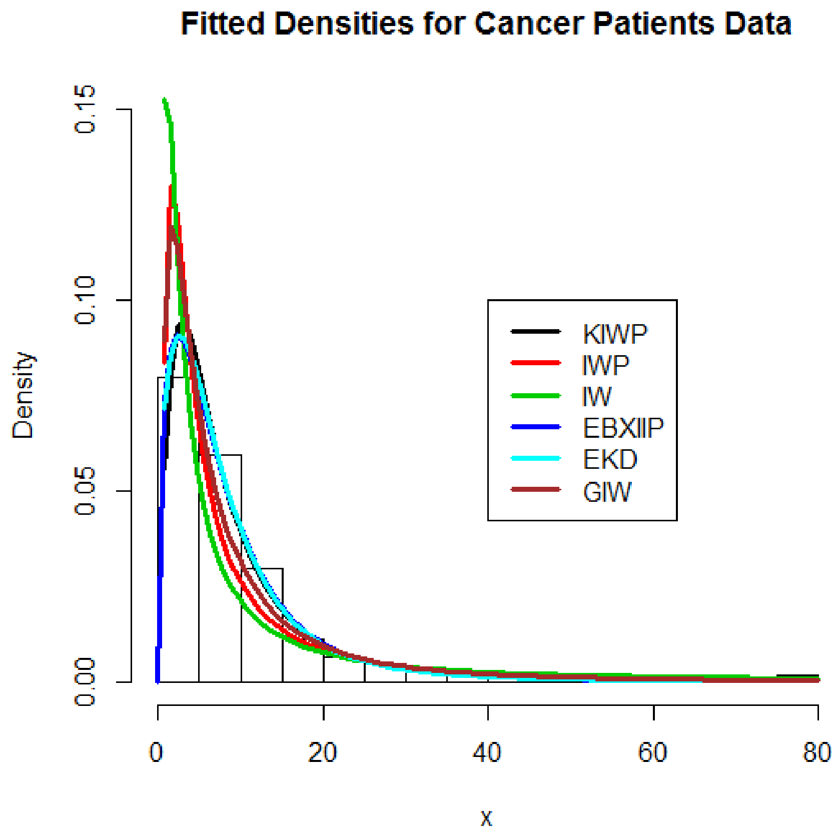

Figure 4. Fitted densities of cancer patients Data

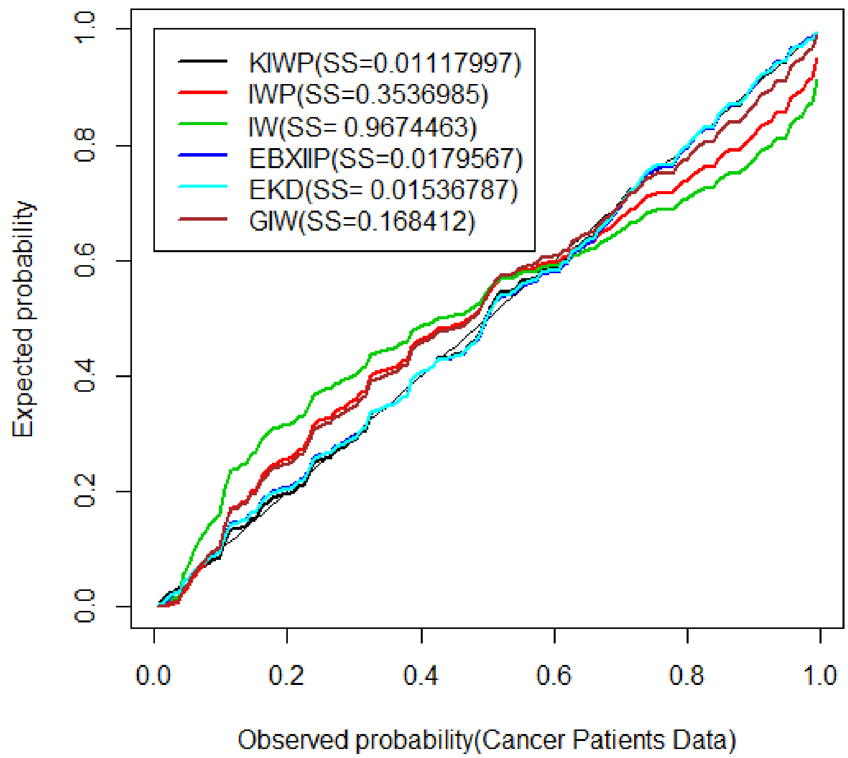

Figure 5. Probability plots of cancer patients data

For the cancer patients data, the likelihood ratio (L.R.) statistics for the test of hypothesis:

$H_0 : IWP(\alpha, \beta.\theta, 1, 1)$ vs $H_a : KIWP(\alpha, \beta, \theta, a, b)$ is $\omega = (854.3 - 817.9) = 36.4$. The corresponding p value $= \chi^2(LL_{IWP} - LL_{KIWP}, 9999, 2) = \chi^2(854.3 - 817.9, 9999, 2) = 1.247 \times 10^{-8} < 0.05$. Consequently, we reject the null hypothesis in favor of the KIWP distribution and we conclude that the KIWP is significantly better than the IWP distribution. For the values of the statistics for the cancer patients data, we note that the KIWP model is better than the IWP, IW, EBXIIP, EKD and GIW models in terms of fitting this particular dataset. The Anderson Darling statistic (A*) and the Cramer von-Mises (W*) goodness of fit statistics are also included in the table to check the data fitting accuracy. The model with the smallest SS, Anderson Darling and Cramer von Mises Statistics gives the best fit for the data thus making the KIWP the best model since $SS = 0.011$, $A^* = 0.074$ and $W^* = 0.012$ which are the smallest values of SS, A* and W* among all the models tested. By inspection, the KIWP is the best model since we observe that the black line on the SS plot is the closest to the straight line.

Guinea Pigs Data Set

The second data set represents the survival times (in days) of 72 guinea pigs infected with virulent tubercle bacilli, observed and reported by Bjerkedal et al. (1960). The starting point of the iterative processes for the guinea pigs data set is $(1.0, 0.009, 10.0, 0.1, 0.1)$.

Table 6. *Survival Times (in days) of Guinea Pigs Infected with Virulent Tubercle Bacilli*

0.1	0.33	0.44	0.56	0.59	0.72	0.74	0.77	0.92	0.93
0.96	1	1	1.02	1.05	1.07	1.07	1.08	1.08	1.08
1.09	1.12	1.13	1.15	1.16	1.2	1.21	1.22	1.22	1.24
1.3	1.34	1.36	1.39	1.44	1.46	1.53	1.59	1.6	1.63
1.63	1.68	1.71	1.72	1.76	1.83	1.95	1.96	1.97	2.02
2.13	2.15	2.16	2.22	2.3	2.31	2.4	2.45	2.51	2.53
2.54	2.54	2.78	2.93	3.27	3.42	3.47	3.61	4.02	4.32
4.58	5.55								

Estimates of the parameters of the KIWP distribution, Akaike information criterion (AIC), consistent Akaike information criterion (AICC), Bayesian information criterion (BIC) are given in table 7 for guinea pigs data. The sum of squares (SS) and goodness of fit statistics W* and A* are also given.

Table 7. *Estimates Of Models For Guinea Pigs Data*

	Estimates					Statistics						
Model	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\theta}$	$\hat{a}$	$\hat{b}$	$-2\log L$	AIC	$AICC$	BIC	SS	A^*	W^*
KIWP$(\alpha,\beta,\theta,a,b)$	0.691	1.417	65.964	0.082	3.454	183.6	193.6	194.5	205.0	0.052	0.343	0.054
	(0.221)	(0.436)	(82.679)	(0.112)	(2.738)							
IWP(α,β,θ)	0.165	1.645	8.321			209.7	215.7	216.0	222.5	0.251	2.098	0.310
	(0.046)	(0.120)	(2.072)									
IW(α,β,θ,a)	1.069	1.173				236.3	240.3	240.5	244.9	0.752	3.835	0.611
	(0.1324)	(0.0843)										
EKum-Dagum$(\alpha,\beta,\theta,a,b)$	1.864	5.997	2.931	1.382	0.452	187.0	197.0	197.9	208.4	1.269	0.415	0.0628
	(7.339)	(7.520)	(2.011)	(1.745)	(1.870)							
GIW(α,β,θ)	140.9	0.132	134.04			197.4	203.4	203.7	210.2	0.131	1.1032	0.151
	( )	(0.000)	( )									
	$\hat{c}$	$\hat{k}$	$\hat{s}$	$\hat{\alpha}$	$\hat{\beta}$							
EBXIIP(c,k,s,α,β)	1.838	7.433	6.708	1.430	2.328	186.9	196.9	197.8	208.2	0.066	0.422	0.063
	(2.574)	(66.794)	(42.709)	(2.258)	(3.164)							

Note. Standard errors are in parentheses

Plots of the fitted densities and histogram, observed probability versus predicted probability for guinea pigs data are given in figures 6 and 7.

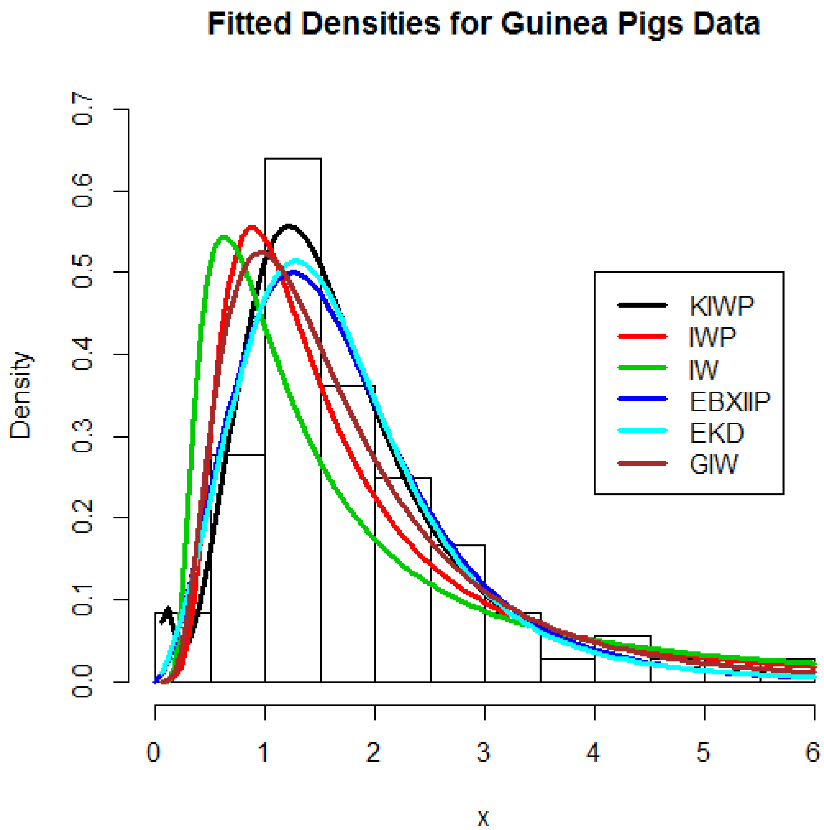

Figure 6. Fitted densities of guinea pigs data

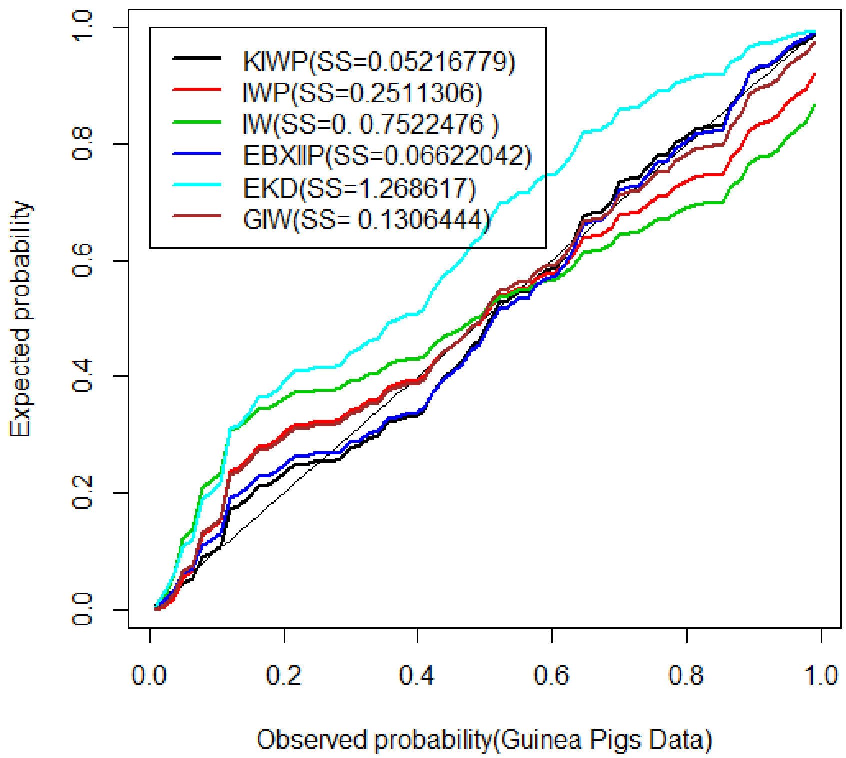

Figure 7. Probability plots of guinea pigs data

For the guinea pigs data, the L.R. statistics for the test of hypothesis: $H_0 : IWP(\alpha, \beta.\theta, 1, 1)$ vs $H_a : KIWP(\alpha, \beta, \theta, a, b)$ is $\omega = (236.3 - 183.6) = 52.7$. The corresponding p value $= \chi^2(LL_{IWP} - LL_{KIWP}, 9999, 2) = \chi^2(236.3 - 183.6, 9999, 2) = 3.600 \times 10^{-12} < 0.05$. Consequently, we reject the null hypothesis in favor of the KIWP distribution and we conclude that the KIWP is significantly better than the IWP distribution. For the values of the statistics for the guinea pigs data, we note that the KIWP model is better than the IWP, IW, EBXIIP, EKD and GIW models in terms of fitting this particular dataset. The Anderson Darling statistic (A*) and the Cramer von-Mises (W*) goodness of fit statistics are also included in the table to check the data fitting accuracy. The model with the smallest Anderson Darling and Cramer von Mises Statistics gives the best fit for the data thus making the KIWP the best model since $SS = 0.0522$, $A^* = 0.3425$ and $W^* = 0.0545$ which are the smallest values of SS, A* and W* among all the models tested. By inspection, the KIWP is the best model since we observe that the black line on the SS plot is the closest to the straight line.

Yarn Specimen Data Set

The third dataset consists of data on the number of cycles of failure for 25 specimens of 100 cm specimens of yarn, tested at a particular strain level by Lawless (1999). The starting points for the iterative processes in the yarn specimens data are $(109.35, 1.2920, 3.6125, 1.0, 1.0)$.

Table 8. *Number of Cycles of Failure for Yarn Specimens*

15	20	38	42	61	76	86	98	121	146
149	157	175	176	180	180	198	220	224	251
264	282	321	325	653					

Estimates of the parameters of the KIWP distribution, Akaike information criterion (AIC), consistent Akaike information criterion (AICC), Bayesian information criterion (BIC) are given in table 9 for yarn specimens data. The sum of squares (SS) and goodness of fit statistics W* and A* are also given.

Table 9. *Estimates Of Models For Yarn Specimens Data*

Model	Estimates					Statistics						
	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\theta}$	$\hat{a}$	$\hat{b}$	$-2\log L$	AIC	$AICC$	BIC	SS	A^*	W^*
KIWP$(\alpha,\beta,\theta,a,b)$	108.87	0.913	15.456	0.294	13.696	302.8	312.8	315.9	318.9	0.028	0.234	0.035
	(159.39)	(0.313)	(10.561)	(0.240)	(20.218)							
IWP(α,β,θ)	109.35	1.292	3.613			313.1	319.1	320.3	322.8	0.169	1.247	0.222
	(94.391)	(0.182)	(1.824)									
IW(α,β,θ,a)	86.441	1.011				317.2	321.2	321.7	323.6	0.257	1.587	0.280
	(50.529)	(0.141)										
EKum-Dagum$(\alpha,\beta,\theta,a,b)$	2.988	730.43	1.116	13.181	0.471	304.6	314.6	317.7	320.7	0.045	0.323	0.051
	(5.728)	(7731.58)	(2.1242)	(73.9329)	(1.0977)							
GIW(α,β,θ)	178.04	0.109	104.87			308.9	314.9	316.0	318.5	0.143	0.863	0.154
	(266.63)	(0.109)	(211.24)									
	$\hat{c}$	$\hat{k}$	$\hat{s}$	$\hat{\alpha}$	$\hat{\beta}$							
EBXIIP(c,k,s,α,β)	0.375	0.644	247.70	6.262	868.09	304.9	314.9	318.0	321.0	0.057	0.385	0.065
	(2.546)	(6.744)	(2415.85)	(65.434)	(5794.65)							

Note. Standard errors are in parentheses

Plots of the fitted densities and histogram, observed probability versus predicted probability for yarn specimens data are given in figures 8 and 9.

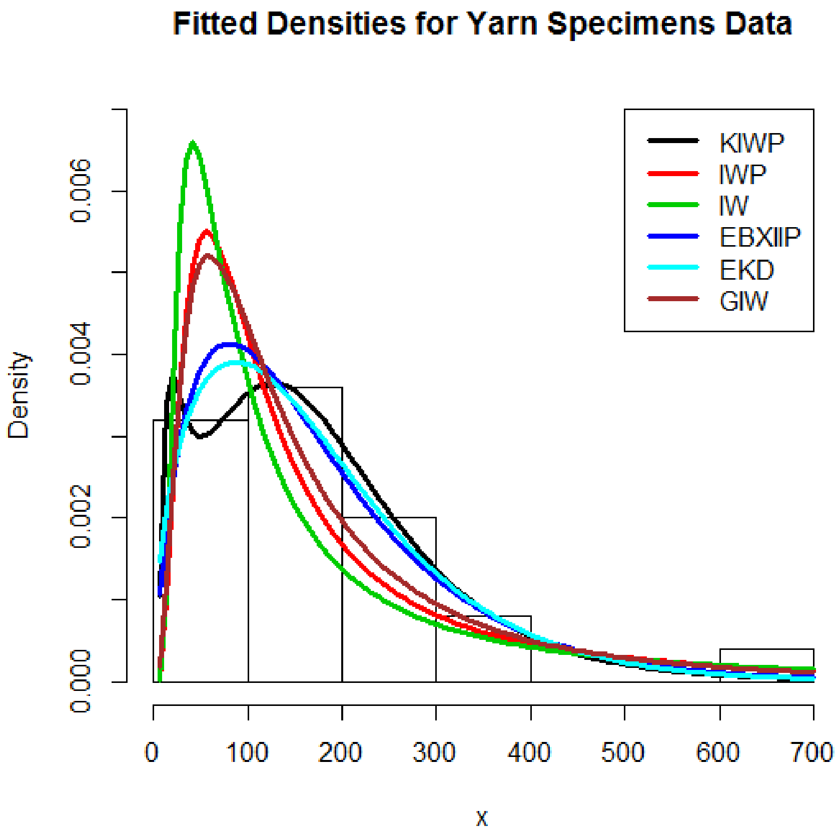

Figure 8. Fitted densities of yarn specimens data

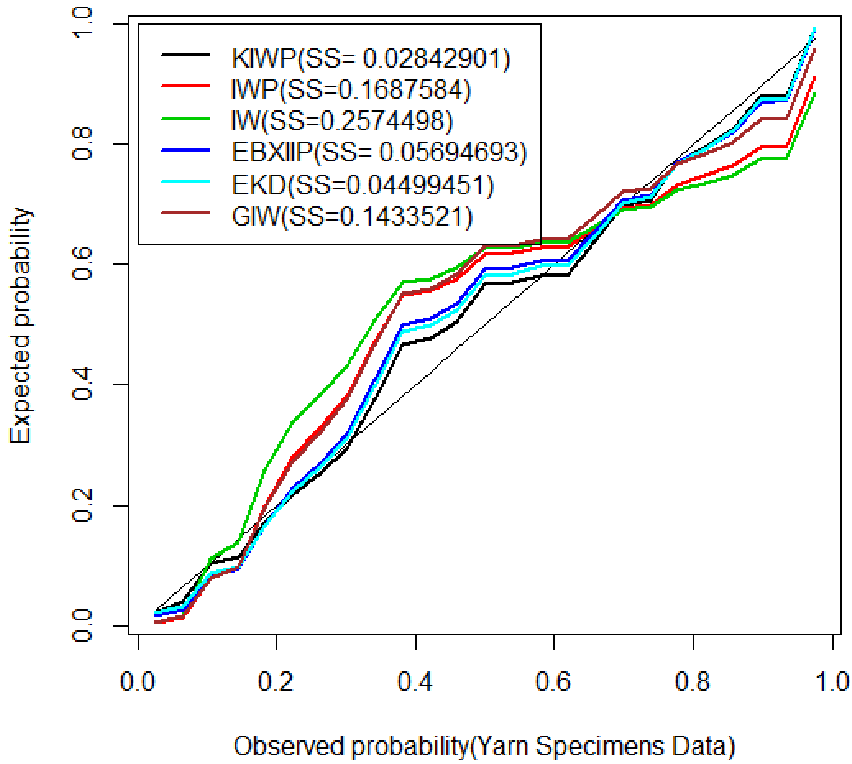

Figure 9. Probability plots of yarn specimen data

For the yarn specimens data, the L.R. statistics for the test of hypothesis: $H_0 : IWP(\alpha, \beta.\theta, 1, 1)$ vs $H_a : KIWP(\alpha, \beta, \theta, a, b)$ are is $\omega = (313.1 - 302.8) = 10.3$. The corresponding p value $= \chi^2(LL_{IWP} - LL_{KIWP}, 9999, 2) = \chi^2(313.1 - 302.8, 9999, 2) = 0.005799 < 0.05$. Consequently, we reject the null hypothesis in favor of the KIWP distribution and we conclude that the KIWP is significantly better than the IWP distribution. For the values of the statistics for the yarn specimens data, we note that the KIWP model is better than the IWP, IW, EBXIIP, EKD and GIW models in terms of fitting this particular dataset. The Anderson Darling statistic (A*) and the Cramer von-Mises (W*) goodness of fit statistics are also included in the table to check the data fitting accuracy. The model with the smallest Anderson Darling and Cramer von Mises Statistics gives the best fit for the data thus making the KIWP the best model since $SS = 0.0284$, $A^* = 0.23418$ and $W^* = 0.0347$

which are the smallest values of SS, A* and W* among all the models tested. By inspection, the KIWP is the best model since we observe that the black line on the SS plot is the closest to the straight line.

Glass Fibers Data Set

The fourth dataset consists of the strengths of 1.5 cm glass fibres measured by the National Physical laboratory in England. The data was previously analyzed by Smith and Naylor (1987). The starting points for the iterative processes in the glass fibers data set is $(0.6365, 1.8769, 5.2630, 1.0, 1.0)$.

Table 10. *Strengths of Glass Fibers measured by the National Physical laboratory*

0.55	0.93	1.25	1.36	1.49	1.52	1.58	1.61	1.64	1.68
1.73	1.81	2.00	0.74	1.04	1.27	1.39	1.49	1.53	1.59
1.61	1.66	1.68	1.76	1.82	2.01	0.77	1.11	1.28	1.42
1.50	1.54	1.60	1.62	1.66	1.69	1.76	1.84	2.24	0.81
1.13	1.29	1.48	1.50	1.55	1.61	1.62	1.66	1.70	1.77
1.84	0.84	1.24	1.30	1.48	1.51	1.55	1.61	1.63	1.67
1.70	1.78	1.89							

Estimates of the parameters of the KIWP distribution, Akaike information criterion (AIC), Consistent Akaike information criterion (AICC), Bayesian information criterion (BIC) are given in table 11 for glass fibers data. The sum of squares (SS) and goodness of fit statistics W* and A* are also given.

Table 11. *Estimates Of Models For Glass Fibers Data*

	Estimates					Statistics						
Model	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\theta}$	$\hat{a}$	$\hat{b}$	$-2\log L$	AIC	$AICC$	BIC	SS	A^*	W^*
KIWP$(\alpha,\beta,\theta,a,b)$	4.817	3.212	47.906	0.123	39.146	21.0	31.0	32.1	41.7	0.069	0.442	0.077
	(0.796)	(0.276)	(19.006)	(0.047)	(42.844)							
IWP(α,β,θ)	0.637	3.877	5.263			75.8	81.8	82.2	88.2	0.855	5.311	0.983
	(0.1824)	(0.3096)	(1.3853)									
IW(α,β,θ,a)	1.969	2.888				93.7	97.7	97.9	102.0	1.246	6.487	1.226
	(0.2485)	(0.2344)										
EKum-Dagum$(\alpha,\beta,\theta,a,b)$	4.931	30.617	5.940	12.504	0.187	29.9	39.9	40.9	50.6	0.239	1.226	0.223
	(1.181)	(27.750)	(1.304)	(10.014)	(0.029)							
GIW(α,β,θ)	136.82	0.340	120.68			59.3	65.3	65.7	71.7	0.739	4.103	0.752
	(119.16)	(0.1545)	(111.61)									
	$\hat{c}$	$\hat{k}$	$\hat{s}$	$\hat{\alpha}$	$\hat{\beta}$							
EBXIIP(c,k,s,α,β)	14.515	69.506	2.879	0.409	5.220	30.0	40.0	41.0	50.7	0.206	1.288	0.235
	(17.161)	(2139.25)	(6.916)	(0.507)	(3.623)							

Note. Standard errors are in parentheses

Plots of the fitted densities and histogram, observed probability versus predicted probability for glass fibers data are given in figures 10 and 11.

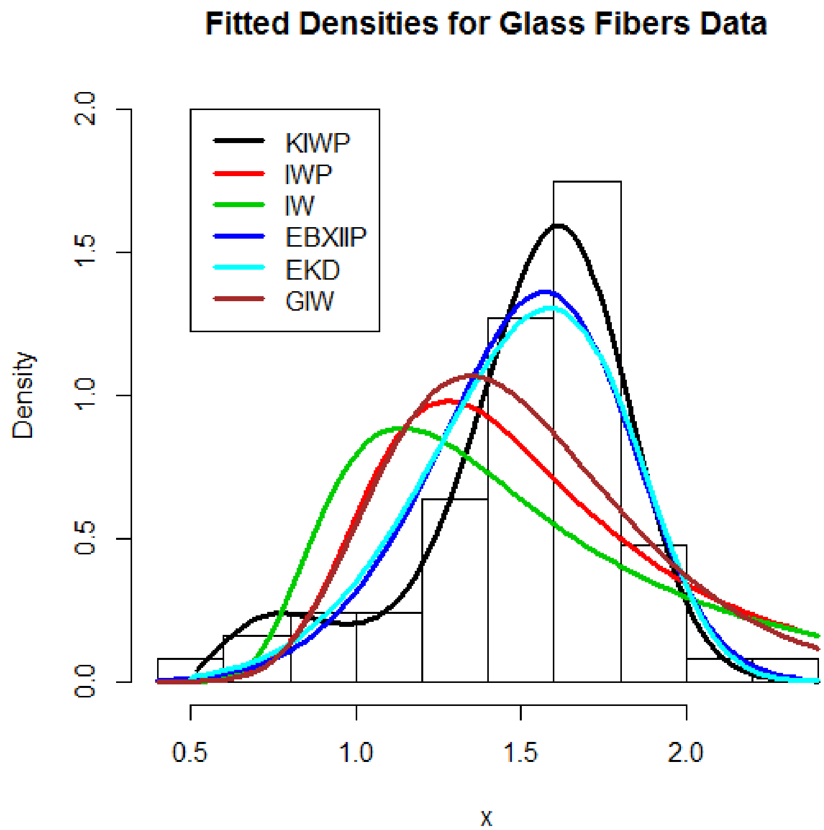

Figure 10. Fitted densities of glass fibers data

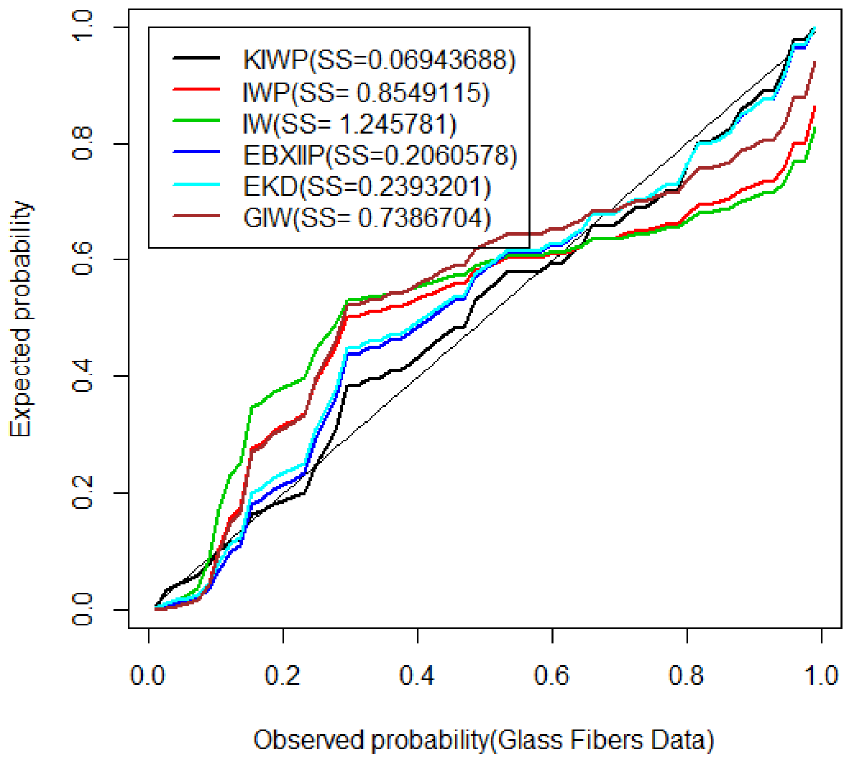

Figure 11. Probability plots of glass fibers data

For the glass fibers data, the L.R. statistics for the test of hypothesis: $H_0 : IWP(\alpha, \beta.\theta, 1, 1)$ vs $H_a : KIWP(\alpha, \beta, \theta, a, b)$ are is $\omega = (75.8 - 21.0) = 54.8$. The corresponding p value $= \chi^2(LL_{IWP} - LL_{KIWP}, 9999, 2) = \chi^2(75.8 - 21.0, 9999, 2) = 1.2599 \times 10^{-12}$. Consequently, we reject the null hypothesis in favor of the KIWP distribution and we conclude that the KIWP is significantly better than the IWP distribution. For the values of the statistics for the glass fibers data, we note that the KIWP model is better than the IWP, IW, EBXIIP, EKD and GIW models in terms of fitting this particular dataset. This is evidenced by the fact that $SS = 0.0694$, $A^* = 0.442062$ and $W^* = 0.07694$ which are the smallest SS, A* and W* values for the glass fibers dataset. By inspection, the KIWP is the best model since we observe that the black line on the SS plot is the closest to the straight line.

Concluding Remarks

In this chapter, we discussed information criteria techniques and the various methods of assessing the best model from a variety of models. We executed a battery of hypothesis tests for the KIWP versus its submodels and further used the sums of squares and plots of the fitted densities to appreciate both numerically and visually the best fitting model. The KIWP was evaluated as the best model after the comparison.

CHAPTER 6

CONCLUSIONS AND SUGGESTIONS FOR FURTHER STUDY

We propose a new class of distribution called the Kumaraswamy inverse Weibull Poisson (KIWP) class of distribution. This class of distribution is obtained by compounding the zero truncated Poisson distribution with the Kumaraswamy inverse Weibull distribution (KIW). The properties of the Kumaraswamy Inverse Weibull Poisson (KIWP) distribution including the hazard function, reverse hazard function, quantile function, moments, survival function, distribution of order statistics, mean deviations, Lorenz and Bonferroni curves and maximum likelihood estimates are presented. We introduce special cases of the KIWP namely the inverse Weibull Poisson distribution (IWP), the Kumaraswamy inverse Rayleigh Poisson distribution (KIRP), Kumaraswamy inverse Exponentiated Poisson (KIEP), Kumaraswamy inverse Frechet Poisson (KIFP) and Kumaraswamy Frechet Poisson (KFP). In addition to the properties of the new class of distribution, reliability and some measures of uncertainty of the KIWP are obtained. The hazard function of the KIWP has different shapes including monotonically increasing, monotone decreasing, upside down bathtub shape, bathtub shape and increasing-decreasing . We carry out a simulation study which evaluates the precision and accuracy of the maximum likelihood estimates of the Kumaraswamy inverse Weibull Poisson distribution model parameters. Real data applications are tabulated and presented to illustrate the applicability of the KIWP.

Suggestions for Further Study

We present suggestions for further studies that may be conducted. In this thesis, we explored the distributional properties and applications of the Kumaraswamy inverse Weibull Poisson distribution. Other possible research topics could include the Kumaraswamy exponentiated inverse Weibull Poisson distribution (EIWP), the gamma inverse Weibull Poisson (GIWP) distribution and the beta inverse Weibull Poisson distributon. Other possible research topics include the Kumaraswamy inverse Weibull geometric (KIWG) and the Kumaraswamy inverse Weibull negative binomial (KIWNB). These models will bring with them the tractability of the Kumaraswamy, gamma and beta distributions and this increases applicability to real life situations such as bounded reservoir storage volume problems investigated by Fletcher and Ponnambalam (1996).

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Appendix A

R Algorithms

```
 f1=function(x,alpha,beta,theta,a,b){
 y=(a*b*alpha*beta*theta*x^(-beta-1)*exp(-alpha*x^(-beta))
*exp(theta*exp(-alpha*x^(-beta)))*((exp(theta)-1)^(-1)))
*(((1-exp(theta*exp(-alpha*x^(-beta))))/(1-exp(theta)))^(a-1))
*((1 - ((1-exp(theta*exp(-alpha*x^(-beta))))/(1-exp(theta)))^a)^(b-1))
 return(y)
}


 moment=function(alpha,beta,theta,a,b,r){
f=function(x,alpha,beta,theta,a,b,r)
{(x^r)*(f1(x,alpha,beta,theta,a,b))}
y=integrate(f,lower=0,upper=Inf,subdivisions=100,alpha=alpha,
beta=beta,theta=theta,a=a,b=b,r=r)
return(y)
}
```

MATLAB Code For Graphing KIWP Hazard Function

```
figure
alpha=1; beta=5; theta=2; a=1.2; b=2;
x=0.001:0.001:5;
g=(theta.*alpha.*beta.*x.^(-beta-1).*exp(-alpha.*x.^(-beta))
.*exp(theta.*exp(-alpha.*x.^(-beta)))./(exp(theta)-1);
G=(1-exp(theta.*exp(-alpha.*x.^(-beta))))./(1-exp(theta));
```

```
f=a.*b.*g.*(G.^(a-1)).*(1-G.^a).^(b-1);
F=(1-(1-G.^a).^b);
h=f./(1-F);
plot(x,h,'g','LineWidth',2.0);
xlabel('x');
ylabel('h(x)');
hold on
```

Code for Sum of Squares of Data Sets

```
data<-sort(data)
n=length(data)
######KIWP#########
alpha1=108.87
beta1=0.9134
theta1=15.4557
a1=0.2938
b1=13.6959

#####IWP##########
alpha2=109.35
beta2=1.2920
theta2=3.6125
a2=1
b2=1

######IW#########
alpha3=86.4408
```

```
beta3=1.0111

#######EBXIIP##############
c4=0.3745
k4=0.6438
s4=247.70
alpha4=6.2616
theta4=868.09

#######EKD###############
alpha5=2.9875
beta5=730.43
theta5=1.1163
a5=13.1808
b5=0.4711

#######GIW#############
alpha6=178.04
beta6=0.1094
theta6=104.87

#################################
F_observed<-c(rep(0,n))
m=n+0.25
for (i in 1:n)
{
F_observed[i]<-(i-0.375)/m
```

```
}

require(gsl)
require(stats)

KIWP_cdf <- function(data,alpha,beta,theta,a,b,n) {
F<-c(rep(0,n))
for (i in 1:n)
{
times<-data[i]
aa1=-alpha*times^(-beta)
aa2=theta*exp(aa1)
aa3=1-exp(aa2)
aa4=1-exp(theta)
aa5=aa3/aa4
aa6=1-(1-aa5^a)^b
F[i]<-aa6
}
return(F)
}

F1<-KIWP_cdf(data,alpha1,beta1,theta1,a1,b1,n)
F1
F2<-KIWP_cdf(data,alpha2,beta2,theta2,a2,b2,n)
F2
```

```
SS<-function(F,F_line,n){
SS<-0
for (i in 1:n)
{
SS<-SS+(F[i]-F_line[i])^2
}
return(SS)
}

SS1<-SS(F1,F_observed,n)
SS1
SS2<-SS(F2,F_observed,n)
SS2

####################################

# Define cdf of IW

IW_cdf <- function(data,alpha,beta,n) {
F<-c(rep(0,n))
for (i in 1:n)
{
times<-data[i]
F[i]=exp(-alpha*times^(-beta))
}
return(F)
}
```

```
F3<-IW_cdf(data,alpha3,beta3,n)
F3
SS<-function(F,F_line,n){
SS<-0
for (i in 1:n)
{
SS<-SS+(F[i]-F_line[i])^2
}
return(SS)
}

SS3<-SS(F3,F_observed,n)
SS3
#############################

# Define cdf of EBXIIP


EBXIIP_cdf <- function(data,c,s,k,alpha,theta,n) {
F<-c(rep(0,n))
for (i in 1:n)
{
times<-data[i]
aa1=1+(times/s)^c
aa2=aa1^-k
aa3=(1-aa2)^alpha
```

```
aa4=1-exp(-theta*aa3)
aa5=1-exp(-theta)
F[i]=aa4/aa5
}
return(F)
}

F4<-EBXIIP_cdf(data,c4,s4,k4,alpha4,theta4,n)
F4
SS<-function(F,F_line,n){
SS<-0
for (i in 1:n)
{
SS<-SS+(F[i]-F_line[i])^2
}
return(SS)
}

SS4<-SS(F4,F_observed,n)
SS4
###############################

#Define cdf of EKD



EKD_cdf <- function(data,alpha,beta,theta,a,b,n) {
F<-c(rep(0,n))
```

```
for (i in 1:n)
{
times<-data[i]
aa1=1+beta*times^(-theta)
aa2=aa1^(-alpha)
aa3=(1-aa2)^a
F[i]=(1-aa3)^b
}
return(F)
}

F5<-EKD_cdf(data,alpha5,beta5,theta5,a5,b5,n)
F5
SS<-function(F,F_line,n){
SS<-0
for (i in 1:n)
{
SS<-SS+(F[i]-F_line[i])^2
}
return(SS)
}

SS5<-SS(F5,F_observed,n)
SS5
################################

#Define cdf GIW
```

```
install.packages("zipfR")
library(zipfR)


#Then use the following code
GIW_cdf=function(data,alpha,beta,theta,n){
F<-c(rep(0,n))
for (i in 1:n)
{
times<-data[i]
ft=Igamma(theta,-log(exp(-alpha
*(times**(-beta)))),lower=FALSE,log=FALSE) /gamma(theta)
F[i]<-ft
}
return(F)
}
F6<-GIW_cdf(data,alpha6,beta6,theta6,n)
F6


SS<-function(F,F_line,n){
SS<-0
for (i in 1:n)
{
SS<-SS+(F[i]-F_line[i])^2
}
return(SS)
}
```

```
SS6<-SS(F6,F_observed,n)
SS6

########################################



op <- par(mfrow=c(1,1))
plot(F_observed,F_observed,type='l',
xlab="Observed probability(Yarn Specimens Data)",ylab="Expected probability")
lines(F_observed,F1,col=1,type='l',lwd=2.0)
lines(F_observed,F2,col=2,lwd=2.0)
lines(F_observed,F3,col=3,type='l',lwd=2.0)
lines(F_observed,F4,col=4,lwd=2.0)
lines(F_observed,F5,col=5,lwd=2.0)
lines(F_observed,F6,col="brown",lwd=2.0)



legend(0.0,1.0, # places a legend at the appropriate place
c("KIWP(SS= 0.02842901)","IWP(SS=0.1687584)",
"IW(SS=0.2574498)","EBXIIP(SS= 0.05694693)",
"EKD(SS=0.04499451)","GIW(SS=0.1433521)"), # puts text in the legend
lty=c(1,1,1,1,1,1), # gives the legend appropriate symbols (lines)
lwd=c(2.0,2.0,2.0,2.0,2.0),col=c(1,2,3,4,5,"brown"))
 # gives the legend lines the correct color and width



#observed vs expected
```

R Code for Plotting Cancer Patients Histogram

```
KIWP<-function(x,alpha,beta,theta,a,b)          {aa<-exp(theta)-1
                                                 ab<-1-exp(theta)
                                                 ac<-x^(-beta-1)
                                                 ad<-exp(-alpha*(x^(-beta)))
                                                 ae<-exp(theta*ad)
                                                 gg1<-(1-ae)/ab
                                                 aa1<-a*b*alpha*beta*theta
                                                 *ac*ad*ae/aa
                                                 ft<-aa1*(gg1^(a-1))
                                                 *((1-gg1^a)^(b-1))
                                                 return(ft)}

IWP<-function(x,alpha,beta,theta)                {aa<-exp(theta)-1
                                                 ab<-1-exp(theta)
                                                 ac<-x^(-beta-1)
                                                 ad<-exp(-alpha*(x^(-beta)))
                                                 ae<-exp(theta*ad)
                                                 gg1<-(1-ae)/ab
                                                 aa1<-1*1*alpha*beta*theta
                                                 *ac*ad*ae/aa
                                                 ft<-aa1*(gg1^(1-1))
                                                 *((1-gg1^1)^(1-1))
                                                 return(ft)}
```

```
IW<-function(x,alpha,beta)                  {aa<-1*1*alpha*beta
                                             *(x^(-beta-1))
                                             ab<-exp(-1*alpha*(x^(-beta)))
                                             ac<-(1-ab)^(1-1)
                                             ft<-aa*ab*ac
                                            return(ft)}

EBXIIP<-function(x,c,k,s,alpha,theta)       {aa<-1+(x/s)^c
                                            aa1<-aa^(-k-1)
                                            aa2<-(1-aa^(-k))^(alpha-1)
                                            aa3<-(1-aa^(-k))^alpha
                                            aa4<-exp(-theta*aa3)
                                            aa5<-c*k*alpha*theta*(x^(c-1))
                                            aa6<-(s^c)*( 1-exp(-theta))
                                            ft<-aa5*aa1*aa2*aa4/aa6
                                            return(ft)}

EKD<-function(x,alpha,beta,theta,a,b)
                                           {aa1<-alpha*beta*theta
                                            *a*b*(x^(-theta-1))
                                            aa2<-1+beta*(x^(-theta))
                                            aa3<-aa1*(aa2^(-alpha-1))
                                            aa4<-1-aa2^(-alpha)
                                            bb1<-aa4^(a-1)
                                            bb2<-aa4^a
```

```
                                   bb3<-(1-bb2)^(b-1)
                                   ft<-aa3*bb1*bb3
                                    return(ft)}

GIW<-function(x,alpha,beta,theta)  {aa<-beta*(x^(-1))/gamma(theta)
                                   ab<-alpha*(x^(-beta))
                                   ac<-ab^(theta)
                                   ad<-exp(-alpha*x^(-beta))
                                   ft<-aa*ac*ad
                                   return(ft)}

hist(x, prob=TRUE, br=15, main='Fitted Densities for Cancer Patients Data'
,ylim=c(0,0.15))
curve(KIWP(x,alpha=1.9009,beta=0.7347,theta=20.5273,a=0.2565,b=5.0406),
lty=1, col=1, add=TRUE,lwd=3)
par(new=TRUE)
curve(IWP(x,alpha=0.6360,beta=1.0023,theta=6.4813),
lty=1, col=2, add=TRUE,lwd=3)
par(new=TRUE)
curve(IW(x,alpha=2.4311,beta=0.7521),lty=1, col=3, add=TRUE,lwd=3)
par(new=TRUE)
curve(EBXIIP(x,c=1.3859,k=0.2138,s=16.6328,alpha=1.0068,theta=14.5423),
lty=1, col=4, add=TRUE,lwd=3)
par(new=TRUE)
curve(EKD(x,alpha=1.6842,beta=43.5588,theta=1.6689,a=1.5826,b=0.4795)
,lty=1, col=5, add=TRUE,lwd=3)
```

```
par(new=TRUE)
curve(GIW(x,alpha=81.3738,beta=0.1107,theta=67.5056),
lty=1, col="brown", add=TRUE,lwd=3)
par(new=TRUE)


legend(40, 0.1, c("KIWP","IWP","IW","EBXIIP","EKD","GIW"),
col = c(1,2,3,4,5,"brown"),
lty = c(1,1,1,1,1,1,1),lwd=c(3,3,3,3,3,3,3))
```

Appendix B

Code for Simulation Study

```
library(numDeriv)

library(Matrix)



alpha=3.5

beta=3.0

theta=4.0

a=0.5

b=0.5


n=50

N=5000



mle_alpha<-c(rep(0,N))

mle_beta<-c(rep(0,N))

mle_theta<-c(rep(0,N))

mle_a<-c(rep(0,N))

mle_b<-c(rep(0,N))



LC_alpha<-c(rep(0,N))

UC_alpha<-c(rep(0,N))
```

```
LC_beta<-c(rep(0,N))

UC_beta<-c(rep(0,N))

LC_theta<-c(rep(0,N))

UC_theta<-c(rep(0,N))

LC_a<-c(rep(0,N))

UC_a<-c(rep(0,N))

LC_b<-c(rep(0,N))

UC_b<-c(rep(0,N))


count_alpha=0

count_beta=0

count_theta=0

count_a=0

count_b=0


temp=1

HH1<-matrix(c(rep(2,25)),nrow=5,ncol=5)

HH2<-matrix(c(rep(2,25)),nrow=5,ncol=5)

for (i in 1:N)

{

print(i)

flush.console()

repeat{

x<-c(rep(0,n))


#Generate a random variable from uniform distribution

u<-0
```

```
u<-runif(n,min=0,max=1)


for (q in 1:n)
{
x[q]<-((-1/alpha)*log((1/theta)*log(1-(1-exp(theta))
*(1-(1-u[q])^(1/b))^(1/a))))^(-1/beta)
}

KIWP_LL<-function(par){-sum(log((par[4]*par[5]*par[1]*par[2]*par[3]
*(x**(-par[2]-1))*(exp(-par[1]*(x**(-par[2]))))*(exp(par[3]*(exp(-par[1]
*(x**(-par[2]))))))/(exp(par[3])-1))*(((1-(exp(par[3]*(exp(-par[1]
*(x**(-par[2])))))))/(1-exp(par[3])))**(par[4]-1))*((1-((1-(exp(par[3]
*(exp(-par[1]*(x**(-par[2])))))))/(1-exp(par[3])))**par[4])**(par[5]-1))))}
#Maximum likelihood estimation
mle.result<-nlminb(c(alpha,beta,theta,a,b),KIWP_LL,lower=0,upper=Inf)

temp=mle.result$convergence
if(temp==0){
temp_alpha<-mle.result$par[1]
temp_beta<-mle.result$par[2]
temp_theta<-mle.result$par[3]
temp_a<-mle.result$par[4]
temp_b<-mle.result$par[5]

HH1<-hessian(KIWP_LL,c(temp_alpha,temp_beta,temp_theta,temp_a,temp_b))
if(is.nan(rcond(HH1)) ==FALSE & rcond(HH1)>1e-06 & sum(is.nan(HH1))==0
```

```
& (diag(HH1)[1]>0)
& (diag(HH1)[2]>0) & (diag(HH1)[3]>0) & (diag(HH1)[4]>0)
& (diag(HH1)[5]>0) )
{
HH2<-solve(HH1)
#print(det(HH1))
}
else{
temp=1}
}

if ((temp==0) & (diag(HH2)[1]>0) & (diag(HH2)[2]>0) & (diag(HH2)[3]>0)
& (diag(HH2)[4]>0)  & (diag(HH2)[5]>0) & (sum(is.nan(HH2))==0)
& sum(is.nan(HH1))==0){
break
}
else{
temp=1
}
}
#print(temp)
temp=1
mle_alpha[i]<-mle.result$par[1]
mle_beta[i]<-mle.result$par[2]
mle_theta[i]<-mle.result$par[3]
mle_a[i]<-mle.result$par[4]
mle_b[i]<-mle.result$par[5]
```

```
HH<-hessian(KIWP_LL,c(mle_alpha[i],mle_beta[i],mle_theta[i],
mle_a[i],mle_b[i]))
H<-solve(HH)

LC_alpha[i]<-mle_alpha[i]-qnorm(0.975)*sqrt(diag(H)[1])
UC_alpha[i]<-mle_alpha[i]+qnorm(0.975)*sqrt(diag(H)[1])
if( (LC_alpha[i]<=alpha) & (alpha<=UC_alpha[i])){
count_alpha=count_alpha+1
}

LC_beta[i]<-mle_beta[i]-qnorm(0.975)*sqrt(diag(H)[2])
UC_beta[i]<-mle_beta[i]+qnorm(0.975)*sqrt(diag(H)[2])
if( (LC_beta[i]<=beta) & (beta<=UC_beta[i])){
count_beta=count_beta+1
}

LC_theta[i]<-mle_theta[i]-qnorm(0.975)*sqrt(diag(H)[3])
UC_theta[i]<-mle_theta[i]+qnorm(0.975)*sqrt(diag(H)[3])
if( (LC_theta[i]<=theta) & (theta<=UC_theta[i])){
count_theta=count_theta+1
}

LC_a[i]<-mle_a[i]-qnorm(0.975)*sqrt(diag(H)[4])
UC_a[i]<-mle_a[i]+qnorm(0.975)*sqrt(diag(H)[4])
if( (LC_a[i]<=a) & (a<=UC_a[i])){
```

```
count_a=count_a+1
}

LC_b[i]<-mle_b[i]-qnorm(0.975)*sqrt(diag(H)[5])
UC_b[i]<-mle_b[i]+qnorm(0.975)*sqrt(diag(H)[5])
if( (LC_b[i]<=b) & (b<=UC_b[i])){
count_b=count_b+1
}

}

#Calculate Average Bias
ABias_alpha<-sum(mle_alpha-alpha)/i
ABias_beta<-sum(mle_beta-beta)/i
ABias_theta<-sum(mle_theta-theta)/i
ABias_a<-sum(mle_a-a)/i
ABias_b<-sum(mle_b-b)/i

#Calculate RMSE
RMSE_alpha<-sqrt(sum((alpha-mle_alpha)^2)/i)
RMSE_beta<-sqrt(sum((beta-mle_beta)^2)/i)
RMSE_theta<-sqrt(sum((theta-mle_theta)^2)/i)
RMSE_a<-sqrt(sum((a-mle_a)^2)/i)
RMSE_b<-sqrt(sum((b-mle_b)^2)/i)

#Converge Probability
CP_alpha<-count_alpha/i
```

```
CP_beta<-count_beta/i
CP_theta<-count_theta/i
CP_a<-count_a/i
CP_b<-count_b/i

#Average Width
AW_alpha<-sum(abs(UC_alpha-LC_alpha))/i
AW_beta<-sum(abs(UC_beta-LC_beta))/i
AW_theta<-sum(abs(UC_theta-LC_theta))/i
AW_a<-sum(abs(UC_a-LC_a))/i
AW_b<-sum(abs(UC_b-LC_b))/i



print(cbind(ABias_alpha,ABias_beta,ABias_theta,ABias_a,Bias_b))
print(cbind(RMSE_alpha,RMSE_beta,RMSE_theta,RMSE_a,RMSE_b))
print(cbind(CP_alpha,CP_beta,CP_theta,CP_a,CP_b))
print(cbind(AW_alpha,AW_beta,AW_theta,AW_a,AW_b))
```