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The Kth M-ary Partition Function

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THE K^{TH} M–ARY PARTITION FUNCTION

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In this thesis we will review the beginnings of integer partitions. We will further explore past research done on restricted partition functions, specifically those of the m-ary and hyper-m-ary partitions for $m \geq 2$. The main content of this paper explores the k^{th} m–ary partition function. For $m, k \geq 2$, let the k^{th} m–ary partition function, $b_m(k, n)$, be the number of ways we can write a positive integer n as a sum of powers of m using at most k of each power. In this thesis we will classify the monotonicity properties of $b_m(k, n)$ by considering the congruence classes of n and k modulo m, and define a relationship between $b_m(1, n)$ and writing n in base m.

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Laura

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LIST OF SYMBOLS

$p(n)$	The Partition Function of n
$b_m(n)$	The m-ary Partition Function
$B_m(x)$	The Generating Function for $b_m(n)$
$h_m(n)$	The Hyper-m-ary Partition Function
$H_m(x)$	The Generating Function for $h_m(n)$
$b_m(k,n)$	The k^{th} m-ary Partition Function
$G_{m,k}(x)$	The Generating Function for $b_m(k,n)$

CHAPTER 1

INTRODUCTION

An integer partition is a representation of a positive integer, n, as a sum of non-increasing positive integers called parts. For example, below are the integer partitions of $n = 6$.

6	3+1+1+1
5+1	2+2+2
4+2	2+2+1+1
4+1+1	2+1+1+1+1
3+3	1+1+1+1+1+1
3+2+1	

A restricted partition is an integer partition that has conditions placed on the size or the number of its parts. Euler is credited to be the first person to formalize the study of integer partitions [4, 18]. In his investigations in 1748, Euler discovered partition identities called Euler pairs, which are composed of two restricted partition functions that can be proven to have the same value. The first Euler pair is *every number has as many integer partitions into odd parts as into distinct parts*. For example, consider the restricted partitions of 6:

odd parts	distinct parts
5+1	5+1
3+3	6
3+1+1+1	3+2+1
1+1+1+1+1+1	4+2

Notice that there are four partitions in both the lists above. Euler proved this equality is true for all integers n. One way to show this result is by using a bijective proof, a one-to-one process that maps odd parts to distinct parts.

Proof from [4]. Moving from odd parts to distinct parts: Here we must ensure that there are no repeated parts. So if our original partition has multiple copies of a part (like

6=3+1+1+1 above), we can combine a pair of the repeated parts until all parts are distinct.

Now we want to define a mapping from distinct parts to odd parts: Since above we combined two parts of equal size, the resulting distinct part will be even. We can split all the even parts in the partition into two halved parts until all parts are odd. QED

So in our example for $n = 6$ above we have the mappings:

$$\begin{array}{lcl} 5+1 & \longmapsto & 5+1 \\ 3+3 & \longmapsto & 6 \\ 3+1+1+1 & \longmapsto & 3+2+1 \\ 1+1+1+1+1+1 & \longmapsto & 4+2 \end{array}$$

The partition function, denoted $p(n)$, is the number of partitions of n. For a restricted partition, we commonly denote its partition function, $p(n|c)$, which is the number of partitions of n under any specified conditions, c [4]. For example, $p(9|$ parts are powers of $2) = 10$ because we have

8+1	2+2+2+2+1
4+4+1	2+2+2+1+1+1
4+2+2+1	2+2+1+1+1+1+1
4+2+1+1+1	2+1+1+1+1+1+1+1
4+1+1+1+1+1	1+1+1+1+1+1+1+1+1

We call the partitions illustrated above the binary partitions of 9. The binary partition function is more often written as $b_2(9) = 10$. We will see more of this partition function in Chapter 2.

So, rather than writing the Euler pair we proved above as "*Every number has as many integer partitions into odd parts as into distinct parts*" we can express it as the as the equality of two partition functions: For all n, $p(n|$ parts are odd$) = p(n|$ parts are distinct$)$.

TABLE 1.1
Some values of $p(n)$.

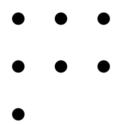
n	$p(n)$	n	$p(n)$	n	$p(n)$	n	$p(n)$
0	1	25	1958	50	204226	75	8118264
1	1	26	2436	51	239943	76	9289091
2	2	27	3010	52	281589	77	10619863
3	3	28	3718	53	329931	78	12132164
4	5	29	4565	54	386155	79	13848650
5	7	30	5604	55	451276	80	15796476
6	11	31	6842	56	526823	81	18004327
7	15	32	8349	57	614154	82	20506255
8	22	33	10143	58	715220	83	23338469
9	30	34	12310	59	831820	84	26543660
10	42	35	14883	60	966467	85	30167357
11	56	36	17977	61	1121505	86	34262962
12	77	37	21637	62	1300156	87	38887673
13	101	38	26015	63	1505499	88	44108109
14	135	39	31185	64	1741630	89	49995925
15	176	40	37338	65	2012558	90	56634173
16	231	41	44583	66	2323520	91	64112359
17	297	42	53174	67	2679689	92	72533807
18	385	43	63261	68	3087735	93	82010177
19	490	44	75175	69	3554345	94	92669720
20	627	45	89134	70	4087968	95	104651419
21	792	46	105558	71	4697205	96	118114304
22	1002	47	124754	72	5392783	97	133230930
23	1255	48	147273	73	6185689	98	150198136
24	1575	49	173525	74	7089500	99	169229875

Finding *p(n)*

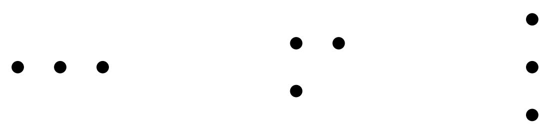
We can observe in Table 1.1 that the partition function $p(n)$ grows quickly. This is to be expected, since there will be more ways to form partitions of n as it gets larger. In fact for all $n \geq 2$,

$$p(n) > p(n-1). \tag{1.1}$$

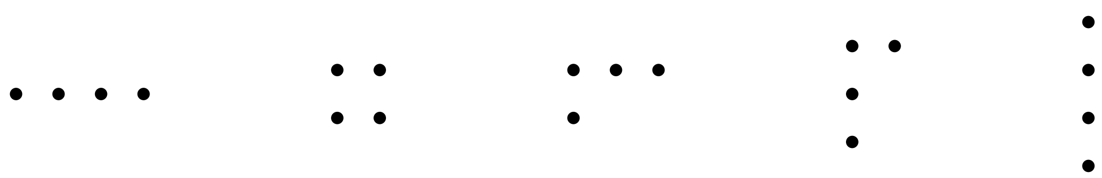
Andrews and Eriksson showed this using Ferrers graphs [4], a representation of an integer partition using a row of m dots to stand for a part of size m. For example, we can write the partition of $7 = 3 + 3 + 1$ as



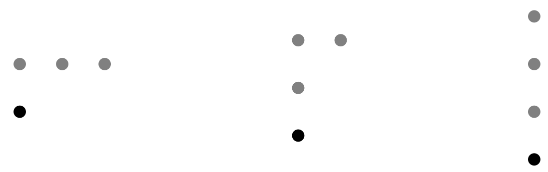
We can illustrate $p(n) > p(n-1)$ for all n, like Andrews and Eriksson, by looking at the Ferrers graphs of the partitions of 3 and 4. We can represent the partitions of three as:



And the Ferrers graphs for the partitions of 4 are:



Comparing these Ferrers graphs we notice that the last three partitions of 4 are obtained from the partitions of 3 by creating a new bottom row with just one dot in it, emphasized below:



So to get back to the partitions of 3, we need only consider the partitions of 4 that have at least one row that is a part of 1 and remove that bottom row. Thus $p(3) = P(4|$ at least one 1-part). In this manner, Andrews and Eriksson proved that for all $n \geq 2$, $p(n-1) = p(n|$ at least one 1-part). So

$$\begin{aligned} p(n) &= p(n|\text{at least one 1-part}) + p(n|\text{no 1-part}) \\ &= p(n-1) + p(n|\text{no 1-part}) \\ &> p(n-1). \end{aligned}$$

Because $p(n)$ is strictly increasing, it would be cumbersome to write out each partition of n every time we want to find the value of $p(n)$. We have some tools that make finding these values much easier: generating functions, recurrences, and asymptotic formulas.

Generating Functions

To quote Herbert Wilf, "A generating function is a clothesline on which we hang up a sequence of numbers for display" [22]. In other words, if we have a sequence of numbers $\{a_0, a_1, a_2, \dots\}$, the generating function, $g(x)$, is a function whose power series expansion is the polynomial where a_n is the coefficient of x^n [4, 22], or

$$g(x) = \sum_{n \geq 0} a_n x^n.$$

Wilf referred to this as a "clothesline" because x is not a variable but rather a placeholder. We call series of this kind formal power series, since we are not concerned with convergence or any other analytic properties of the series.

Viewing $p(n)$ for $n \geq 0$ as a sequence, we can find its generating function

$$g(x) = \sum_{n \geq 0} p(n) x^n$$

by looking at how many of each i-part we could use in a partition of n. Considering only 1-parts:

$$\begin{aligned} &x^0 + x^1 + x^{1+1} + x^{1+1+1} + x^{1+1+1+1} + \cdots \\ &= 1 + x + x^2 + x^3 + x^4 + \cdots \end{aligned} \tag{1.2}$$

where the exponent of x is the number of 1s being added to sum to n and the coefficient of each x^n is the number of ways to get n from summing up n 1s (and there is only one way to do so).

Now for the 2-parts,

$$\begin{aligned} &x^0 + x^2 + x^{2+2} + x^{2+2+2} + x^{2+2+2+2} + \cdots \\ &= 1 + x^2 + x^4 + x^6 + x^8 + \cdots \end{aligned} \tag{1.3}$$

Here the term x^8 is saying there is only one way to use 2s to sum to 8.

Doing the same for the 3-parts,

$$\begin{aligned} &x^0 + x^3 + x^{3+3} + x^{3+3+3} + x^{3+3+3+3} + \cdots \\ &= 1 + x^3 + x^6 + x^9 + x^{12} + \cdots \end{aligned} \tag{1.4}$$

Note that (1.2), (1.3), and (1.4) only consider using exclusively 1's, 2's or 3's, respectively, in the partitions of n. If wanted to use a mixture of those parts, we would multiply

$$(1+x+x^2+x^3+x^4+\cdots)(1+x^2+x^4+x^6+x^8+\cdots)(1+x^3+x^6+x^9+x^{12}+\cdots).$$

So for $p(n)$ with no restrictions on the parts we have

$$\begin{aligned} g(x) &= (1+x+x^2+x^3+x^4+\cdots)(1+x^2+x^4+x^6+x^8+\cdots)\cdots \\ &= \prod_{i\geq 0}(1+x^i+x^{2i}+x^{3i}+\cdots) \\ &= \prod_{i\geq 0}\frac{1}{1-x^i}. \end{aligned} \tag{1.5}$$

Expanding the power series (1.5) we get

$$\begin{aligned} g(x) &= 1+x+2x^2+3x^3+5x^4+7x^5+11x^6+15x^7+22x^8+\cdots \\ &= \sum_{n\geq 0} p(n)x^n. \end{aligned}$$

So (1.5) is the generating function of $p(n)$.

Recall the Euler pair $p(n|$ parts are odd$)$ $=$ $p(n|$ parts are distinct$)$. In the beginning of this chapter we proved this using a bijection, but Euler showed the identity using generating functions [4].

Proof due to Euler, [4] and [10]. The generating function for $p(n|$ parts are distinct$)$ is

$$\begin{aligned} g(x) &= \sum_{n\geq 0} p(n|\text{ parts are distinct})x^n \\ &= (1+x^1)(1+x^2)(1+x^3)(1+x^4)(1+x^5)\cdots \\ &= \left(\frac{1-x^2}{1-x}\right)\left(\frac{1-x^4}{1-x^2}\right)\left(\frac{1-x^6}{1-x^3}\right)\left(\frac{1-x^8}{1-x^4}\right)\left(\frac{1-x^{10}}{1-x^5}\right)\cdots \\ &= \left(\frac{\cancel{1-x^2}}{1-x}\right)\left(\frac{\cancel{1-x^4}}{\cancel{1-x^2}}\right)\left(\frac{\cancel{1-x^6}}{1-x^3}\right)\left(\frac{\cancel{1-x^8}}{\cancel{1-x^4}}\right)\left(\frac{\cancel{1-x^{10}}}{1-x^5}\right)\cdots \end{aligned}$$

Notice that the numerators will cancel out all of the denominators that contain an even power of x. This leaves us with

$$\begin{aligned}
\sum_{n\geq 0} p(n|\text{ parts are distinct})x^n &= \left(\frac{1}{1-x}\right)\left(\frac{1}{1-x^3}\right)\left(\frac{1}{1-x^5}\right)\left(\frac{1}{1-x^7}\right)\cdots \\
&= \prod_{i\geq 0 \text{ odd}} \frac{1}{1-x^i} \\
&= \prod_{i\geq 0 \text{ odd}} (1+x^{2i}+x^{3i}+x^{4i}+\cdots) \\
&= \sum_{n\geq 0} p(n|\text{ parts are odd})x^n.
\end{aligned}$$

From (1.2) we know $1+x^2+x^3+x^4+\cdots$ is the generating function of using only 1s in the partitions of n and from (1.4) the sum $1+x^6+x^9+x^{12}+\cdots$ indicates that only 3's are being used in the partitions of n; so for i odd, the coefficients of $\prod_{i\geq 0 \text{ odd}}(1+x^{2i}+x^{3i}+x^{4i}+\cdots)$ give the number of partitions of n using only odd parts. QED

Recurrences

Another way we can find the n^{th} term of the sequence of $p(n)$ is using a recursive formula. A recurrence is an implicit formula for the n^{th} term of a sequence using some of the preceding terms. For example, the well known Fibonacci sequence

$$\{0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \dots\}$$

has the recurrence $F_n = F_{n-1}+F_{n-2}$ for $n \geq 2$, with the initial values $F_0 = 0$ and $F_1 = 1$.

Euler's work with pentagonal numbers, numbers defined as $n = \frac{j(3j\pm 1)}{2}$ for $j = 0, 1, 2, \dots,$ gave a partition identity that results in a recursion for $p(n)$.

Theorem 1.1. Euler's Pentagonal Number Theorem, from [4, 12]

$$p(n|\ \textit{even number of parts all distinct}) - p(n|\ \textit{odd number of parts all distinct})$$
$$= \begin{cases} (-1)^j & \textit{if } n = j(3j \pm 1)/2 \\ 0 & \textit{otherwise} \end{cases}$$

Proof from [4, 12]. The generating function of $p(n|$ even number of parts all distinct$)$ $-$ $p(n|$ odd number of parts all distinct$)$ is given by:

$$\begin{aligned}&\sum_{n\geq 0} \big(p(n|\text{ even number of parts all distinct}) \\ &\qquad\qquad -p(n|\text{ odd number of parts all distinct})\big)\, x^n \\ &= \prod_{i\geq 0}(1-x^i).\end{aligned}$$

Since this equals 0 when $i \neq j(3j \pm 1)/2$ we get:

$$\begin{aligned}&= \sum_{j\geq 0}(-1)^j x^{j(3j\pm 1)/2} \\ &= \sum_{j\geq 0}(-1)^j x^{j(3j-1)/2} + \sum_{j\geq 1}(-1)^j x^{j(3j+1)/2} \\ &= 1 + \sum_{j\geq 1}(-1)^j x^{j(3j-1)/2} + \sum_{j\geq 1}(-1)^j x^{j(3j+1)/2} \\ &= 1 + \sum_{j\geq 1}(-1)^j (1-x^j) x^{j(3j-1)/2}.\end{aligned}$$

Recall that the generating function of $p(n)$ is

$$\sum_{n\geq 0} p(n)x^n = \prod_{i\geq 0}\frac{1}{1-x^i}.$$

Then rearranging we have

$$\prod_{i\geq 0}(1-x^i)\sum_{n\geq 0}p(n)x^n = 1.$$

Above we found $\prod_{i\geq 0}(1-x^i) = 1+\sum_{i\geq 1}(-1)^j(1-x^j)x^{j(3j-1)/2}$, so substituting this into the above equation, we get

$$\left(1+\sum_{j\geq 1}(-1)^j(1-x^j)x^{j(3j-1)/2}\right)\sum_{n\geq 0}p(n)x^n = 1.$$

Comparing the coefficients of x^n on both sides we get

$$\begin{aligned}&p(n)-p(n-1)-p(n-2)+p(n-5)+p(n-7)-\\&\qquad\cdots+(-1)^j p\left(n-\frac{j(3j-1)}{2}\right)+(-1)^j p\left(n-\frac{j(3j+1)}{2}+\right)\cdots=0.\end{aligned}$$

Rearranging, we get a recurrence for $p(n)$:

$$\begin{aligned}&p(n)=p(n-1)+p(n-2)-p(n-5)-p(n-7)+\\&\qquad\cdots+(-1)^j p\left(n-\frac{j(3j-1)}{2}\right)+(-1)^j p\left(n-\frac{j(3j+1)}{2}+\right)\cdots\end{aligned}$$

where the i in $p(n-i)$ represents a pentagonal number. QED

For example, given $p(0)=1$ we can find:

$$\begin{aligned}p(1)&=p(0)=1\\p(2)&=p(2-1)+p(2-2)=p(1)+p(0)=2\\p(3)&=p(3-1)+p(3-2)=p(2)+p(1)=3\\&\vdots\\p(10)&=p(9)+p(8)-p(5)-p(3)=30+22-7-3=42\\&\vdots\end{aligned}$$

Asymptotic Formulas

For n large, finding $p(n)$ using the generating function or recurrence, although not as tedious as counting by hand, can take a while to calculate. An asymptotic formula, in this context, is an expression that approximates a function of natural numbers, in this case $p(n)$, such that when $n \to \infty$ the ratio of $p(n)$ and the approximation approaches 1. Hardy, Ramanujan, and Rademacher found an exact formula for $p(n)$ [4, 13, 17]:

$$p(n) = \frac{1}{\pi\sqrt{2}} \left[\frac{d}{dx} \frac{\sinh\left(\pi\left(\frac{2}{3}(x - \frac{1}{24})\right)^{1/2}\right)}{(x - \frac{1}{24})^{1/2}} \right]_{x=n} + \text{similar terms}$$

Where the "similar terms" are determined by a 24^{th} root of unity; and, as mentioned in [3, 4], the proof and use of this equation depends on the theory of functions of complex variables. Instead we will consider the first term of the series above which gives an asymptotic formula for $p(n)$

$$p(n) \sim \frac{1}{4n\sqrt{3}} e^{\pi\left(\frac{2n}{3}\right)^{1/2}}.$$

For example, from Table 1.1 we see that $p(10) = 42$. The asymptotic formula yields

$$p(10) \approx \frac{1}{4(10)\sqrt{3}} e^{\pi\left(\frac{2(10)}{3}\right)^{1/2}} = 48.10430882$$

which gives us the ratio:

$$\frac{42}{48.10430882} = 0.8731026603.$$

The number of partitions of 99 given in Table 1.1 is $p(99) = 169,229,875$. Now the asymptotic formula gives us

$$p(99) \approx \frac{1}{4(99)\sqrt{3}} e^{\pi\left(\frac{2(99)}{3}\right)^{1/2}} = 177,006,854.2$$

which gives us the ratio:

$$\frac{169,229,875}{177,006,854.2} = 0.9560639658$$

And to take an example from [3], $p(200) = 3,972,999,029,388$ while the asymptotic formula for $n = 200$ is

$$p(200) \approx \frac{1}{4(200)\sqrt{3}} e^{\pi\left(\frac{2(200)}{3}\right)^{1/2}} = 4,100,251,432,187.8$$

and our ratio is

$$\frac{3,972,999,029,388}{4,100,251,432,187.8} = 0.9689647318.$$

Observe that as $n \to \infty$ the ratio of the true value of $p(n)$ to the approximation is approaching 1.

Congruences

As Hardy and Ramanujan were developing the exact equation for $p(n)$ in the last section, Ramanujan noticed while looking at the table of $p(n)$, organized like our Table 1.1, that the last value of every group of five was a multiple of 5, or

$$p(5n + 4) \equiv 0(\text{mod } 5). \tag{1.6}$$

He later found two other simple congruences:

$$p(7n + 5) \equiv 0(\text{mod } 7) \tag{1.7}$$

and

$$p(11n + 6) \equiv 0(\text{mod } 11). \tag{1.8}$$

Notice how the coefficients of n in (1.6), (1.7), and (1.8), are consecutive primes. So one would think that there are more congruences of the form $p(bn + c) \equiv 0(\text{mod } b)$ where b is prime and c is any positive integer. But it has been shown that the three congruences above are the only ones of this form. These three congruences are referred to today as the Ramanujan Congruences [4, 13]. Ramanujan also conjectured similar congruences using 5^i and 7^i, specifically:

$$p(5^2m + 24) \equiv 0(\text{mod } 5^2)$$

$$p(7^2m + 47) \equiv 0(\text{mod } 7^2)$$

Congruences are greatly valued in the study of integer partitions. Although they do not give the value of a partition function, congruences reveal much about the arithmetic properties of the partition function being studied, which are usually very hard to discern [14].

CHAPTER 2

REVIEW OF PAST RESEARCH

The Binary Partition

For an integer n, the binary partition function, $b_2(n)$, is the number of representations of n as a sum of powers of 2, i.e. the number of ways n can be represented as

$$n = \sum_{i=0}^{\infty} c_i 2^i, \text{ where } c_i \in \{0, 1, 2, 3, \ldots\}.$$

One Euler pair in particular originated the binary partition: *the number of partitions of n with parts of size 1 is equal to the number of partitions of n whose parts are distinct powers of* 2 [4, 10], which can be written as

$$p(n| \text{ parts in } \{1\}) = p(n| \text{ parts are distinct powers of } 2). \tag{2.1}$$

Proof from Andrews and Eriksson [4]. Since the only way to represent an integer n with parts being restricted to 1 is simply writing n as a sum of n ones, we have $p(n| \text{ parts in } \{1\}) = 1$. We will merge pairs of ones into twos, then merge the pairs of twos into fours, and continue to combine pairs of numbers until all the parts are distinct.

Now, starting with partitions of n where the parts must be distinct powers of two, we can employ what Andrews and Eriksson call a "splitting process" (what we did above they called a "merging process"). So we can take each of the distinct powers of two and divide them by 2 until we have all ones.

Thus we have a bijection that shows

$$p(n| \text{ parts in } \{1\}) = p(n| \text{ parts are distinct powers of } 2\ ). \qquad \text{QED}$$

Euler had also defined $b_2(n)$ the binary partition function; which unlike $p(n|$ parts are distinct powers of 2$)$, does not restrict the number of each power of two that can be included in the partitions.

Recall the example from Chapter 1, $p(9|$ parts are powers of 2 $) = b_2(9) = 10$, since we have the following binary partitions of 9:

8+1	2+2+2+2+1
4+4+1	2+2+2+1+1+1
4+2+2+1	2+2+1+1+1+1+1
4+2+1+1+1	2+1+1+1+1+1+1+1
4+1+1+1+1+1	1+1+1+1+1+1+1+1+1

Euler calculated some values of $b_2(n)$ (see Table 2.1) using its generating function

$$\begin{aligned} B_2(x) &= \sum_{n\geq 0} b_2(n)x^n \\ &= \prod_{i\geq 0} \frac{1}{1-x^{2^i}}. \end{aligned} \tag{2.2}$$

In 1969 Churchhouse used a computer to calculate $b_2(n)$ for $n \leq 200$ and published the following conjecture on his observations:

$$b_2(2^{2k+2}t) - b_2(2^{2k}t) \equiv 0(\text{mod } 2^{3k+2}) \text{ and } b_2(2^{2k+1}t) - b_2(2^{2k-1}t) \equiv 0(\text{mod } 2^{3k}).$$

He was the first mathematician to find congruence properties of the binary partition function [18, 19]. For some examples of Churchhouse's conjectures, see Tables 2.2 and 2.3. Observe that if we have $k = 2$ and $t = 5$:

$$\begin{aligned} b_2(2^{2k+2}t) - b_2(2^{2k}t) &= b_2(320) - b_2(80) \\ &= 2197788 - 4124 \\ &= 2193664 \equiv 0(\text{mod } 2^8) \end{aligned}$$

TABLE 2.1
Some Values of $b_2(n)$.

n	$b_2(n)$	n	$b_2(n)$	n	$b_2(n)$	n	$b_2(n)$
0	1	25	94	50	786	75	3074
1	1	26	114	51	786	76	3404
2	2	27	114	52	900	77	3404
3	2	28	140	53	900	78	3734
4	4	29	140	54	1014	79	3734
5	4	30	166	55	1014	80	4124
6	6	31	166	56	1154	81	4124
7	6	32	202	57	1154	82	4514
8	10	33	202	58	1294	83	4514
9	10	34	238	59	1294	84	4964
10	14	35	238	60	1460	85	4964
11	14	36	284	61	1460	86	5414
12	20	37	284	62	1626	87	5414
13	20	38	330	63	1626	88	5938
14	26	39	330	64	1828	89	5938
15	26	40	390	65	1828	90	6462
16	36	41	390	66	2030	91	6462
17	36	42	450	67	2030	92	7060
18	46	43	450	68	2268	93	7060
19	46	44	524	69	2268	94	7658
20	60	45	524	70	2506	95	7658
21	60	46	598	71	2506	96	8350
22	74	47	598	72	2790	97	8350
23	74	48	692	73	2790	98	9042
24	94	49	692	74	3074	99	9042

TABLE 2.2
Some Values of $b_2(2^{2k+2}t) - b_2(2^{2k}t)$.

$k \setminus t$	1	3	5
1	32	672	4064
2	1792	168704	2193664
3	690176	357378048	10599626752
4	2319826944	7354598342656	516453041848320

TABLE 2.3
Some Values of $b_2(2^{2k+1}t) - b_2(2^{2k-1}t)$.

$k \setminus t$	1	3	5
1	8	88	376
2	192	8256	72512
3	27136	5885440	114931200
4	30224384	38339547136	1743048527872

and

$$
\begin{aligned}
b_2(2^{2k+1}t) - b_2(2^{2k-1}t) &= b_2(160) - b_2(40) \\
&= 72902 - 390 \\
&= 72512 \equiv 0(\text{mod } 2^6).
\end{aligned}
$$

Churchhouse's conjectures were proven first by Rødseth, followed by three alternate proofs by Gupta [9, 19].

Theorem 2.1. From Churchhouse, [9]
If $k \geq 1$ and $t \equiv 1(mod\ 2)$, then

$$b_2(2^{2k+2}t) - b_2(2^{2k}t) \equiv 0(mod\ 2^{3k+2})$$

and

$$b_2(2^{2k+1}t) - b_2(2^{2k-1}t) \equiv 0(mod\ 2^{3k}).$$

Hirschhorn and Loxton gave a generalization for a class of like congruences of $b_2(n)$ [15, 18].

Theorem 2.2. From Hirschhorn and Loxton, [15]
Let $\lambda(n) = 1$ or -1 if the binary expansion of n ends in an even or odd number of zeros, respectively. And let $\mu(n) = 1$ or -1 depending on the number of ones being even or odd in the binary expansion of n. Then the binary partition function satisfies:

- $b_2(2n) \equiv 1 + \lambda(n)(mod\ 4), \quad (n \geq 1)$
- $b_2(4n+2) \equiv 2\mu(n)(mod\ 8)$
- $b_2(8n) \equiv \mu(n)\left(1 - 3\lambda(n)\right)(mod\ 8), \quad (n \geq 1)$
- $b_2(8n+2) \equiv 2\mu(n)(mod\ 16)$

- $b_2(8n+4) \equiv 4(\textit{mod}\ 8)$

- $b_2(8n+6) \equiv 6\mu(n)(\textit{mod}\ 16)$

- $b_2(2^{2r}n + 2^{2r-2}) \equiv 4\mu(n)(\textit{mod}\ 32), \quad (r \geq 2)$

- $b_2(2^{2r}n + 3 \cdot 2^{2r-2}) \equiv 20\mu(n)(\textit{mod}\ 32), \quad (r \geq 2)$

- $b_2(2^{2r+1}n + 2^{2r-1}) \equiv 10\mu(n)(\textit{mod}\ 16), \quad (r \geq 2)$

- $b_2(2^{2r+1}n + 3 \cdot 2^{2r-1}) \equiv 14\mu(n)(\textit{mod}\ 16), \quad (r \geq 2)$

These congruences, together with those which follow from the equation $b_2(2n+1) = b_2(2n)$, are the only congruences of the form $b_2(2^r n + t) \equiv X(n)(\textit{mod}\ 2^k)$, where $0 \leq t < r^r$. Also, $X(n)$ is a function that depends on n only through $\lambda(n)$ or $\mu(n)$.

The *m*-ary Partition

For an integer n, the m-ary partition function, $b_m(n)$, is the number of representations of n as a sum of powers of m, i.e. the number of representations of the form:

$$n = \sum_{i=0}^{\infty} c_i m^i, \text{ where } c_i \in \mathbb{N}.$$

For example, $b_4(16) = 6$ since there are six partitions of 16 that use only powers of 4 as parts:

16	4+4+1+1+1+1+1+1+1+1
4+4+4+4	4+1+1+1+1+1+1+1+1+1+1+1+1
4+4+4+1+1+1+1	1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1

To find the generating function of $b_m(n)$, we simply need to replace the 2 in $B_2(x)$ with m:

$$\begin{aligned} B_m(x) &= \sum_{n\geq 0} b_m(n)x^n \\ &= \prod_{i\geq 0} \frac{1}{1-x^{m^i}}. \end{aligned}$$

We can use this to calculate some values of $b_m(n)$ for $3 \leq m \leq 10$ (see Table 2.4).

Following the Churchhouse conjectures, Andrews generalized congruences for partitions using powers of m rather than powers of 2.

Theorem 2.3. From Andrews, [1]

$$b_m(m^{r+1}n) - b_m(m^r n) \equiv 0(\textit{mod } \mu^r)$$

where $\mu = m$ if m is odd and $\mu = \frac{m}{2}$ if m is even.

Table 2.5 gives us some values that we can use to illustrate Theorem 2.3. For example, for $r = 2, m = 3$, and $n = 5$ we have

$$\begin{aligned} b_3(3^{2+1}(5)) - b_3(3^2(5)) &= 954 - 63 \\ &= 891 \\ &\equiv 0(\text{mod } 3^2). \end{aligned}$$

In 2001 Rødseth and Sellers showed that the congruence above, along with the Churchhouse conjectures, are just a subset of a larger family of congruences.

TABLE 2.4
Some Values of $b_m(n)$.

$n \setminus m$	3	4	5	6	7	8	9	10
0	1	1	1	1	1	1	1	1
1	1	1	1	1	1	1	1	1
2	1	1	1	1	1	1	1	1
3	2	1	1	1	1	1	1	1
4	2	2	1	1	1	1	1	1
5	2	2	2	1	1	1	1	1
6	3	2	2	2	1	1	1	1
7	3	2	2	2	2	1	1	1
8	3	3	2	2	2	2	1	1
9	5	3	2	2	2	2	2	1
10	5	3	3	2	2	2	2	2
11	5	3	3	2	2	2	2	2
12	7	4	3	3	2	2	2	2
13	7	4	3	3	2	2	2	2
14	7	4	3	3	3	2	2	2
15	9	4	4	3	3	2	2	2
16	9	6	4	3	3	3	2	2
17	9	6	4	3	3	3	2	2
18	12	6	4	4	3	3	3	2
19	12	6	4	4	3	3	3	2
20	12	8	5	4	3	3	3	3
21	15	8	5	4	4	3	3	3
22	15	8	5	4	4	3	3	3
23	15	8	5	4	4	3	3	3
24	18	10	5	5	4	4	3	3
25	18	10	7	5	4	4	3	3
26	18	10	7	5	4	4	3	3
27	23	10	7	5	4	4	4	3
28	23	12	7	5	5	4	4	3
29	23	12	7	5	5	4	4	3
30	28	12	9	6	5	4	4	4

TABLE 2.5
Some Values of $b_3(3^r n)$.

$n \setminus r$	1	2	3	4	5
1	2	5	23	239	5828
2	3	12	93	1632	68457
3	5	23	239	5828	342383
4	7	40	508	15439	1150492
5	9	63	954	34488	3082437

Theorem 2.4. From Rødseth and Sellers, [20]

Let $r \geq 1$ and suppose that σ_r can be expressed as

$$\sigma_r = \sum_{i=2}^{r} \epsilon_i m^i$$

where $\epsilon_i \in \{0,1\}$ for each i. Finally, let $c_r = 1$ if m is odd, and let $c_r = 2^{r-1}$ if m is even. Then, for all $n \geq 1$,

$$b_m(m^{r+1}n - \sigma_r - m) \equiv 0 \left(mod \ \frac{m^r}{c_r} \right).$$

This is equivalent to Andrews congruence when $\sigma_r = 0$

The Hyper-*m*-ary Partiton

Thus far we have looked at partitions where the size of the parts have been restricted to powers of $m \geq 2$. We can further explore the m-ary partions by now looking to restrict the number of each part we may use. The hyper-m-ary partition, which is the writing of an integer n as a sum of powers of m where each power can be used at most m times.

For example, $h_3(10) = 2$ because the hyper 3-ary partitions of 10 are

9+1

and

3+3+3+1.

The hyper m-ary partition function has the generating function

$$\begin{aligned} H_m(x) &= \sum_{n \geq 0} h_m(n) x^n \\ &= \prod_{i \geq 0} \frac{1 - x^{(m+1)m^i}}{1 - x^{m^i}} \end{aligned}$$

which we can use to find the values in Table 2.6.

TABLE 2.6
Some Values of $h_m(n)$.

$n \setminus m$	2	3	4	5	6	7	8	9	10
0	1	1	1	1	1	1	1	1	1
1	1	1	1	1	1	1	1	1	1
2	2	1	1	1	1	1	1	1	1
3	1	2	1	1	1	1	1	1	1
4	3	1	2	1	1	1	1	1	1
5	2	1	1	2	1	1	1	1	1
6	3	2	1	1	2	1	1	1	1
7	1	1	1	1	1	2	1	1	1
8	4	1	2	1	1	1	2	1	1
9	3	3	1	1	1	1	1	2	1
10	5	2	1	2	1	1	1	1	2
11	2	2	1	1	1	1	1	1	1
12	5	3	2	1	2	1	1	1	1
13	3	1	1	1	1	1	1	1	1
14	4	1	1	1	1	2	1	1	1
15	1	2	1	2	1	1	1	1	1
16	5	1	3	1	1	1	2	1	1
17	4	1	2	1	1	1	1	1	1
18	7	3	2	1	2	1	1	2	1
19	3	2	2	1	1	1	1	1	1
20	8	2	3	2	1	1	1	1	2
21	5	3	1	1	1	2	1	1	1
22	7	1	1	1	1	1	1	1	1
23	2	1	1	1	1	1	1	1	1
24	7	2	2	1	2	1	2	1	1
25	5	1	1	3	1	1	1	1	1
26	8	1	1	2	1	1	1	1	1
27	3	4	1	2	1	1	1	2	1
28	7	3	2	2	1	2	1	1	1
29	4	3	1	2	1	1	1	1	1
30	5	5	1	3	2	1	1	1	2

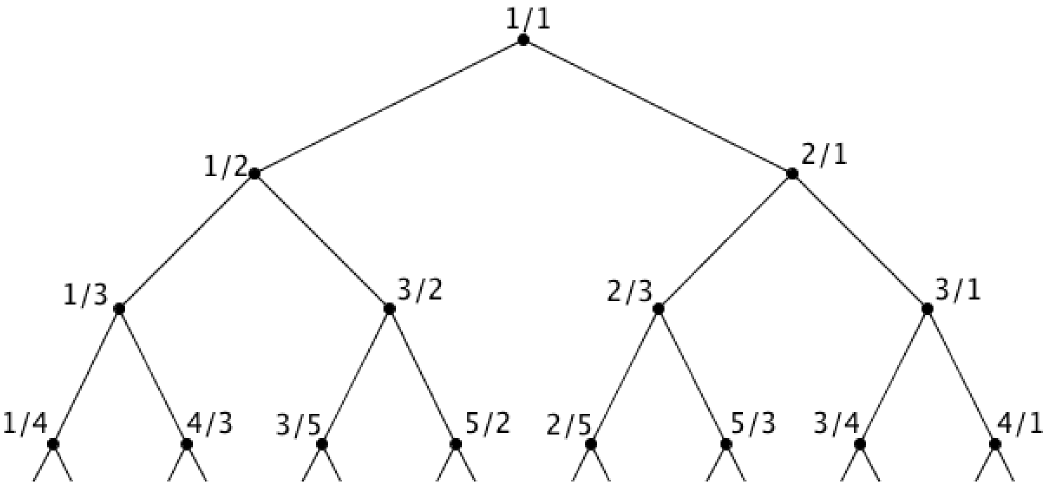


FIG 2.1. *Calkin and Wilf Tree.*

Naturally exploration of the hyper-m-ary partition function, $h_m(n)$, began with the case $m = 2$.

The sequence of the values of the hyperbinary partition function has been known since the 1860's (although not in reference to partitions) as the Stern-Brocot sequence [6, 21]. Calkin and Wilf found a recurrences for the hyperbinary partition function in their alternative proof of the countability of the rational numbers [7]. They constructed a tree of all rational numbers under two rules:

(1) First, $\frac{1}{1}$ is the topmost vertex of the tree, and

(2) Every vertex $\frac{i}{j}$ has two children: $\frac{i}{i+j}$ on the left and $\frac{i+j}{j}$ on the right.

See Figure 2.1.

This will give the list of positive rational numbers as:

$$\frac{1}{1}, \frac{1}{2}, \frac{2}{1}, \frac{1}{3}, \frac{3}{2}, \frac{2}{3}, \frac{3}{1}, \frac{1}{4}, \frac{4}{3}, \frac{3}{5}, \ldots$$

Calkin and Wilf noted that having the rationals listed in this way makes the sequence of the numerators the values of $h_2(n)$ for $n \geq 0$, and since the denominator of the

n^{th} fraction is the same as the numerator of the following, each fraction is $\frac{h_2(n)}{h_2(n+1)}$ [7]. Using their definition of the left and right children of each vertex gives the recurrences for $n \geq 0$ where $h_2(0) = 1$:

$$h_2(2n+1) = h_2(n)$$

and

$$h_2(2n+2) = h_2(n) + h_2(n+1).$$

Now using these recurrences, Courtright and Sellers found a congruence for the hyperbinary partition function, $h_2(n)$, described in the following theorm.

Theorem 2.5. From Courtright and Sellers, [8]
For all $n \geq 0$,

$$h_2(3n+2) \equiv 0 (\textit{mod } 2).$$

They also found recurrences for the hyper-m-ary partition function

$$h_m(mn) = h_m(n) + h_m(n-1)$$

$$h_m(mn+r) = h_m(n)$$

for $1 \leq r \leq m-1$. They then used these recurrences to show the following result.

Theorem 2.6. From Courtright and Sellers, [8]
Let $m \geq 3$ and $j \geq 1$ be fixed integers, and let k be some integer between 2 and $m-1$. Then for all $n \geq 0$,

$$h_m(m^i n + m^{j-1}k) = j h_m(n).$$

Other Restrictions on *m*-ary Partitions

Where the hyper-m-ary partitions allow at most m of each power in the representation of n, Rødseth and Sellers [20] defined restricted m-ary partition function, $b_{m,k}(n)$, which counts the partitions of n of the form

$$n = \sum_{i\geq 0} c_i m^i \text{ where } c_i \in \{0, 1, 2, \ldots, m^k - 1\}.$$

The generating function for $b_{k,m}(n)$ is

$$\begin{aligned} B_{m,k}(x) &= \sum_{n\geq 0} b_{m,k}(n)x^n \\ &= \prod_{i=0}^{k-1} \frac{1}{1-q^{m^i}}. \end{aligned}$$

They used this restricted m-ary partition function to prove Theorem 2.4 by way of proving the following stronger result.

Theorem 2.7. From Rødseth and Sellers, [20]

Let $r \geq 1, k \geq 2$, and $s = \min(r, k-1)$. Moreover, let σ_s and c_s be defined as in Theorem 2.4. Then

$$b_{m,k}(m^{r+1}n - \sigma_s - m) \equiv 0 \left(\mathit{mod}\ \frac{m^r}{c_s} \right).$$

Reznick presented the d^{th} binary partition in [18], which restricts the binary partition by allowing at most $d-1$ of each power of 2, by writing an integer n as:

$$n = \sum_{i=0}^{\infty} c_i 2^i, \text{ where } c_i \in \{0, 1, 2, \ldots, d-1\}.$$

For the purposes of this paper we will view this as the k^{th} binary partition, where we can use at most k of each power of 2 (where $k+1=d$):

$$n = \sum_{i=0}^{\infty} c_i 2^i, \text{ where } c_i \in \{0, 1, 2, \dots, k\}.$$

The number of such partitions is denoted by $b_2(k,n)$. We have already seen partition functions of this form, for example, the partition from 2.1,

$$p(n|\text{parts are distinct powers of 2}) = b_2(1,n) = 1. \tag{2.3}$$

We can also write the hyperbinary partition function as

$$h_2(n) = b_2(2,n), \tag{2.4}$$

and Reznick denotes the binary partitions of n as

$$b_2(n) = b_2(\infty, n).$$

He also notes that another value for $b_2(k,n)$ is also known from the 1983 Putnam Exam [16]

$$b_2(3,n) = \left\lfloor \frac{n}{2} \right\rfloor + 1 \tag{2.5}$$

which later we will generalize for all $m \geq 2$.

Reznick found the generating function of $b_2(k,n)$ to be

$$\begin{aligned} G_{2,k}(x) &= \sum_{n \geq 0} b_2(k,n) x^n \\ &= \prod_{i=0}^{\infty} \frac{1 - x^{(k+1)2^i}}{1 - x^{2^i}}. \end{aligned} \tag{2.6}$$

Using this and the parity of k and n, Reznick described the behavior of the sequences of $b_2(k, n)$ in the following result.

Theorem 2.8. From Reznick, [18]

When we have k odd and n even:

$$\text{i) } b_2(k, n) = b_2(k, n+1)$$
$$\text{ii) } b_2(k, n) > b_2(k, n-1)$$

And for k and n both even:

$$\text{iii) } b_2(k, n) > b_2(k, n-1)$$
$$\text{iv) } b_2(k, n) \geq b_2(k, n+1)$$

This can be observed in Table 2.7 for the values $0 \leq n \leq 30$ and $1 \leq k \leq 10$.

To see Theorem 2.8 i) notice that looking down all the columns of Table 2.7 that when k is odd we have pairs of repeated values for $b_2(k, n)$ where the first of the pair occurs when n is even. We can see Theorem 2.8 ii) in a similar way, but now we compare the value of one pair to that of the previous pair. Parts i) and ii) of the theorem show, the nondecreasing behavior of $b_2(k, n)$ when k is odd. Now looking at the pairs down the columns when k is even, we can instantly see that $b_2(k, n)$ increases between the pairs, but remains the same or decreases within a pair.

Theorem 2.8 iv) is especially interesting, since the values of all the partition functions mentioned thus far, except for $h_m(n)$, have been at least nondecreasing, if not strictly increasing (see Tables 1.1, 2.1, 2.4, and 2.5). This behavior makes sense due to the fact that as n gets larger, the number of integers that we can use in a partition of n gets larger, and thus the number of partitions of n will also grow. But now when the extra restriction is placed on the number of parts of $b_2(n)$ we see a decrease in the number of partitions between $b_2(k, n)$ and $b_2(k, n+1)$. Also notice as k grows the sequence of

$b_2(k, n)$ is matching $b_2(n)$ in an increasing number of terms, which is fitting of Reznick's notation of $b_2(\infty, n)$ for the binary partition (see Tables 2.1 and 2.7).

In this paper we will describe similar monotonicity properties for $b_m(k, n)$ by considering congruence classes of n and k modulo m.

TABLE 2.7
Some Values of $b_2(k, n)$.

$n \setminus k$	1	2	3	4	5	6	7	8	9	10
0	1	1	1	1	1	1	1	1	1	1
1	1	1	1	1	1	1	1	1	1	1
2	1	2	2	2	2	2	2	2	2	2
3	1	1	2	2	2	2	2	2	2	2
4	1	3	3	4	4	4	4	4	4	4
5	1	2	3	3	4	4	4	4	4	4
6	1	3	4	5	5	6	6	6	6	6
7	1	1	4	4	5	5	6	6	6	6
8	1	4	5	8	8	9	9	10	10	10
9	1	3	5	6	8	8	9	9	10	10
10	1	5	6	9	10	12	12	13	13	14
11	1	2	6	7	10	10	12	12	13	13
12	1	5	7	12	13	16	16	18	18	19
13	1	3	7	8	13	14	16	16	18	18
14	1	4	8	12	14	19	20	22	22	24
15	1	1	8	9	14	15	20	20	22	22
16	1	5	9	17	18	24	25	30	30	32
17	1	4	9	12	18	20	25	26	30	30
18	1	7	10	18	21	28	30	35	36	40
19	1	3	10	14	21	22	30	31	36	36
20	1	8	11	23	26	34	36	44	45	50
21	1	5	11	15	26	29	36	38	45	46
22	1	7	12	22	28	39	42	50	52	59
23	1	2	12	16	28	30	42	44	52	53
24	1	7	13	28	33	46	49	62	64	72
25	1	5	13	19	33	38	49	52	64	66
26	1	8	14	27	36	52	56	68	72	84
27	1	3	14	20	36	40	56	59	72	74
28	1	7	15	32	40	59	64	81	84	98
29	1	4	15	20	40	49	64	68	84	88
30	1	5	16	29	41	64	72	88	93	110

CHAPTER 3

THE K^{th} M–ARY PARTITION FUNCTION

The Generating Function of $b_m(k,n)$

For $m, k \geq 2$, the k^{th} m–ary partition function, denoted $b_m(k,n)$, is the number of ways we can write a positive integer n as a sum of powers of m using at most k of each power. In other words $b_m(k,n)$ is the number of representations of the form

$$n = \sum_{i \geq 0} c_i m^i \quad \text{where } c_i \in \{0, 1, \ldots, k\}.$$

We can find the generating function of $b_m(k,n)$ as follows:

$$\begin{aligned} G_{m,k}(x) &= \sum_{i \geq 0} b_m(k,n) x^n \\ &= \prod_{i \geq 0} (1 + x^{m^i} + x^{2m^i} + \cdots + x^{km^i}) \\ &= \prod_{i \geq 0} \frac{1 - x^{(k+1)m^i}}{1 - x^{m^i}}. \end{aligned}$$

Just as Reznick [18] presented the known values of $b_2(k,n)$ in (2.3), (2.4) and (2.5), we can use the generating function of $b_m(k,n)$ above to generalize these equations for all m.

Theorem 3.1. *For $b_m(k,n)$ we have*

a) $b_m(m-1, n) = 1$

b) $b_m(m, n) = h_m(n)$

c) $b_m(m^2 - 1, n) = \lfloor \frac{n}{m} \rfloor + 1.$

Proof of Theorem 3.1 a.

$$\begin{aligned}
G_{m,m-1}(x) &= \prod_{i\geq 0} \frac{1-x^{(m-1+1)m^i}}{1-x^{m^i}} \\
&= \prod_{i\geq 0} \frac{1-x^{m^{i+1}}}{1-x^{m^i}} \\
&= \frac{1-x^m}{1-x} \cdot \frac{1-x^{m^2}}{1-x^m} \cdot \frac{1-x^{m^3}}{1-x^{m^2}} \cdot \frac{1-x^{m^4}}{1-x^{m^3}} \cdots \\
&= \frac{1}{1-x} \cdot \prod_{i\geq 0} \frac{1-x^{m^{i+1}}}{1-x^{m^{i+1}}} \\
&= \frac{1}{1-x} \\
&= 1 + x + x^2 + x^3 + \cdots
\end{aligned}$$

Extracting the coefficients of x^n for all $n \geq 0$ yields

$$b_m(m-1,n) = 1.$$

QED

Note the $(m-1)^{st}$ m-ary partition of n is equivalent to writing n in base m, so this proof guarantees that there exists a unique way to write any integer in base m.

Proof of Theorem 3.1 b. We know that $b_m(m,n)$ is the number of ways we can write n as a sum of powers m using at most m of each part, and $h_m(n)$ also counts those partitions. So by definition, $b_m(m,n) = h_m(n)$. QED

Proof of Theorem 3.1 c.

$$\begin{aligned}
G_{m,m^2-1}(x) &= \prod_{i\geq 0} \frac{1-x^{(m^2-1+1)m^i}}{1-x^{m^i}} \\
&= \prod_{i\geq 0} \frac{1-x^{m^{i+2}}}{1-x^{m^i}}
\end{aligned}$$

$$= \frac{1-x^{m^2}}{1-x} \cdot \frac{1-x^{m^3}}{1-x^m} \cdot \frac{1-x^{m^4}}{1-x^{m^2}} \cdot \frac{1-x^{m^5}}{1-x^{m^3}} \cdot \frac{1-x^{m^6}}{1-x^{m^4}} \cdots$$
$$= \frac{1}{1-x} \cdot \frac{1}{1-x^m} \cdot \prod_{i\geq 0} \frac{1-x^{m^{i+2}}}{1-x^{m^{i+2}}}$$
$$= \frac{1}{1-x} \cdot \frac{1}{1-x^m}$$
$$= (1+x+x^2+x^3+\cdots)(1+x^m+x^{2m}+x^{3m}+\cdots)$$
$$= 1+x+x^2+x^3+\cdots+2x^m+2x^{m+1}+\cdots+3x^{2m}+3x^{2m+1}+\cdots$$

Extracting the coefficients of x^n for all $n \geq 0$ gives us a sequence of m ones, followed by m twos, then m threes and so on. The formula for such a sequence is $\lfloor \frac{n}{m} \rfloor + 1$. Thus

$$b_m(m^2-1,n) = \left\lfloor \frac{n}{m} \right\rfloor + 1.$$

QED

Monotonicity of *b*$_m$*(k,n)*

We wish to generalize Reznick's results in [18] on the monotonic nature of $b_m(k,n)$, which we labeled Theorem 2.8.

Observe in Table 3.1 of values for $b_3(k,n)$ where $k \equiv 0 (\mathrm{mod}\ 3)$ that at some point, the first number of the triplet is greater than those following. Notice that this is true for n being any multiple of $k \equiv 0 (\mathrm{mod}\ 3)$. But this does not describe every n where $b_3(k,n) > b_3(k,n+1)$. In fact for all $n \geq k$, we see this pattern of decreasing every subsequent multiple of 3.

To be precise, for $1 \leq k \leq n$ such that $k, n \equiv 0 (\mathrm{mod}\ 3)$ we have

$$b_3(k,n) > b_3(k,n+1). \tag{3.1}$$

Table 3.1

Some Values of $b_3(k, n)$.

$n \setminus k$	1	2	3	4	5	6	7	8	9	10
0	1	1	1	1	1	1	1	1	1	1
1	1	1	1	1	1	1	1	1	1	1
2	0	1	1	1	1	1	1	1	1	1
3	1	1	2	2	2	2	2	2	2	2
4	1	1	1	2	2	2	2	2	2	2
5	0	1	1	1	2	2	2	2	2	2
6	0	1	2	2	2	3	3	3	3	3
7	0	1	1	2	2	2	3	3	3	3
8	0	1	1	1	2	2	2	3	3	3
9	1	1	3	3	3	4	4	4	5	5
10	1	1	2	3	3	3	4	4	4	5
11	0	1	2	2	3	3	3	4	4	4
12	1	1	3	4	4	5	5	5	6	6
13	1	1	1	4	4	4	5	5	5	6
14	0	1	1	2	4	4	4	5	5	5
15	0	1	2	3	4	6	6	6	7	7
16	0	1	1	3	4	4	6	6	6	7
17	0	1	1	1	4	4	4	6	6	6
18	0	1	3	3	4	7	7	7	9	9
19	0	1	2	3	4	5	7	7	7	9
20	0	1	2	2	4	5	5	7	7	7
21	0	1	3	4	4	7	8	8	10	10
22	0	1	1	4	4	5	8	8	8	10
23	0	1	1	2	4	5	6	8	8	8
24	0	1	2	3	4	7	8	9	11	11
25	0	1	1	3	4	4	8	9	9	11
26	0	1	1	1	4	4	5	9	9	9
27	1	1	4	4	5	8	9	10	14	14
28	0	1	3	4	5	6	9	10	11	14
29	1	1	3	3	5	6	6	10	11	11
30	1	1	5	6	6	9	10	11	15	16

TABLE 3.2

Some Values of $b_4(k, n)$.

$n \setminus k$	1	2	3	4	5	6	7	8	9	10
0	1	1	1	1	1	1	1	1	1	1
1	1	1	1	1	1	1	1	1	1	1
2	0	1	1	1	1	1	1	1	1	1
3	0	0	1	1	1	1	1	1	1	1
4	1	1	1	2	2	2	2	2	2	2
5	1	1	1	1	2	2	2	2	2	2
6	0	1	1	1	1	2	2	2	2	2
7	0	0	1	1	1	1	2	2	2	2
8	0	1	1	2	2	2	2	3	3	3
9	0	1	1	1	2	2	2	2	3	3
10	0	1	1	1	1	2	2	2	2	3
11	0	0	1	1	1	1	2	2	2	2
12	0	0	1	2	2	2	2	3	3	3
13	0	0	1	1	2	2	2	2	3	3
14	0	0	1	1	1	2	2	2	2	3
15	0	0	1	1	1	1	2	2	2	2
16	1	1	1	3	3	3	3	4	4	4
17	1	1	1	2	3	3	3	3	4	4
18	0	1	1	2	2	3	3	3	3	4
19	0	0	1	2	2	2	3	3	3	3
20	1	1	1	3	4	4	4	5	5	5
21	1	1	1	1	4	4	4	4	5	5
22	0	1	1	1	2	4	4	4	4	5
23	0	0	1	1	2	2	4	4	4	4
24	0	1	1	2	3	4	4	6	6	6
25	0	1	1	1	3	4	4	4	6	6
26	0	1	1	1	1	4	4	4	4	6
27	0	0	1	1	1	2	4	4	4	4
28	0	0	1	2	2	3	4	6	6	6
29	0	0	1	1	2	3	4	4	6	6
30	0	0	1	1	1	3	4	4	4	6

From the values in the columns of Table 3.1 where $k \equiv 1 (\text{mod } 3)$, we can see that (3.1) also appears to be true if we have $n \equiv 1 (\text{mod } 3)$.

Next, in the columns where $k \equiv 2 (\text{mod } 3)$, observe that $b_3(k, n)$ will never decrease and notice that for some n it increases. More formally, if $k \equiv 2 (\text{mod } 3)$, then

$$b_3(k, n) \leq b_3(k, n + 1). \tag{3.2}$$

We can see the patterns above reappearing for $m > 3$ as well. For example, see Table 3.2. When $n, k \equiv 0 (\text{mod } 4)$, $n, k \equiv 1 (\text{mod } 4)$, or $n, k \equiv 2 (\text{mod } 4)$ note that $b_4(k, n) > b_4(k, n + 1)$; and when $n, k \equiv 3 (\text{mod } 4)$ we have that $b_4(k, n) \leq b_4(k, n + 1)$.

These observations lead to the following main results of this thesis.

Theorem 3.2. *Let $m, n, k \in \mathbb{N}$ such that $1 \leq k \leq n$ and $m \geq 3$. If $k \equiv n (\text{mod } m)$ and $n \not\equiv m - 1 (\text{mod } m)$, then*

$$b_m(k, n) > b_m(k, n + 1).$$

Theorem 3.3. *Let $m, n, k \in \mathbb{N}$ such that $1 \leq k \leq n$ and $m \geq 3$. If $k \equiv m - 1 (\text{mod } m)$ and $n \not\equiv m - 1 (\text{mod } m)$, then*

$$b_m(k, n) = b_m(k, n + 1).$$

Some Essential Lemmas

To verify Theorem 3.2 we will need three lemmas. First we will examine how we can associate the m-ary partitions of n to the m-ary partitions of $n + 1$ given that $n \not\equiv m - 1 (\text{mod } m)$.

Consider the 3-ary partitions of $n = 10 \equiv 1 (\text{mod } 3)$ and the 3-ary partitions of $n + 1 = 11$:

$n = 10$	$n + 1 = 11$
9+1	9+1+1
3+3+3+1	3+3+3+1+1
3+3+1+1+1+1	3+3+1+1+1+1+1
3+1+1+1+1+1+1+1	3+1+1+1+1+1+1+1+1
1+1+1+1+1+1+1+1+1+1	1+1+1+1+1+1+1+1+1+1+1

Observe that the 3-ary partitions of 11 result from adding 1 (shown in gray) to the 3-ary partitions of 10. This leads to Lemma 3.4.

Lemma 3.4. *Let $m, n, \in \mathbb{N}$. If $n \not\equiv m - 1 (\text{mod } m)$, then adding 1 to the m-ary partitions of n yields the m-ary partitions of $n + 1$.*

Proof. We can write the m-ary partitions of n as

$$n = \sum_{i \geq 0} c_i m^i \qquad \text{for } c_i \in \mathbb{N}.$$

Let m^j be the largest power of m such that $m^j \leq n$. Then $c_i = 0$ for all $i > j$. So now we can represent all the partitions of n as

$$n = c_j m^j + c_{j-1} m^{j-1} + \cdots + c_2 m^2 + c_1 m^1 + c_0 m^0. \tag{3.3}$$

Then by adding 1 to each side of the equation above we get

$$n + 1 = c_j m^j + c_{j-1} m^{j-1} + \cdots + c_2 m^2 + c_1 m^1 + (c_0 + 1) m^0.$$

Note that the above encompasses all the m-ary partitions of $n+1$, because adding any additional copy of m^i for $i \leq j$ other than m^0 to the partitions of n will cause the resulting sum to be greater than $n+1$. Further, since $n \not\equiv m-1 (\text{mod } m)$, the current number of ones, $c_0 + 1$ is not a multiple of m and cannot be combined to create an additional copy of m^i.

Also note that none of the m-ary partitions of $n+1$ will contain the next largest power of m, m^{j+1} since

$$\begin{aligned} n &\leq m^j \\ n+1 &\leq m^j+1 \\ &< m^{j+1}. \end{aligned}$$

QED

Next we consider the structure of all the m-ary partitions of $n+1$ for $n \not\equiv m-1 (\text{mod } m)$. Specifically, we are interested in the minimum quantity of 1-parts that must appear in the partitions of n.

Again, consider the 3-ary partitions of $n = 10 \equiv 1(\text{mod } 3)$ and the 3-ary partitions of $n+1 = 11$:

$n = 10$	$n+1 = 11$
9+1	9+1+1
3+3+3+1	3+3+3+1+1
3+3+1+1+1+1	3+3+1+1+1+1+1
3+1+1+1+1+1+1+1	3+1+1+1+1+1+1+1+1
1+1+1+1+1+1+1+1+1+1	1+1+1+1+1+1+1+1+1+1+1

Notice in this example $n \mod m = 1$ and that every partition of 11 has at least $1+1=2$ 1-parts. This brings us to Lemma 3.5.

Lemma 3.5. *Let $n \not\equiv m-1(\text{mod } m)$ and let $a = n \mod m$ for $m, n \in \mathbb{N}$. Then every m-ary partition of $n+1$ has at least $a+1$ ones.*

Proof. Suppose by way of contradiction that there exists a partition of $n+1$ that has fewer than $a+1$ ones. Then we could write that partition as

$$n+1 = \sum_{i \geq 0} c_i m^i = \sum_{i \geq 1} c_i m^i + c_0 m^0 \quad \text{for } c_i \in \mathbb{N}.$$

Now since this these partitions have less than $a + 1$ ones, $c_0 \leq a$; and note that since $n \equiv a(\text{mod } m)$, by modular arithmetic we must have $n + 1 \equiv a + 1(\text{mod } m)$. But with $c_0 \leq a$ the partitions above must be congruent to $a(\text{mod } m)$, $a - 1(\text{mod } m)$, $\dots$, or $0(\text{mod } m)$; which is a contradiction. QED

Lastly, we consider the k^{th} m-ary partitions of n. Specifically, we verify that there will be at least one partition in which we have the maximum amount of 1's. This is crucial in proving the main result.

Lemma 3.6. *Let $m, n, k \in \mathbb{N}$. If $k \equiv n(\text{mod } m)$ and $n \not\equiv m - 1(\text{mod } m)$. Then at least one k^{th} m-ary partition of n has exactly k 1's.*

Proof. Let $a = n \mod m$. Consider the k^{th} m-ary partitions of n as written in equation (3.3) now with the restriction on the c_i's

$$n = c_j m^j + c_{j-1} m^{j-1} + \cdots + c_2 m^2 + c_1 m^1 + c_0 m^0 \quad \text{where } c_i \in \{0, 1, \dots, k\}.$$

Suppose that in all the partitions above $c_0 < k$. We will perform an algorithm that transforms a partition like those above into one with exactly k 1's.

Since $n \equiv a(\text{mod } m)$ we must have $c_0 \equiv a(\text{mod } m)$. Also by properties of modular arithmetic $k - c_0 \equiv 0(\text{mod } m)$.

$(*)$ Let m^s, $0 < s \leq j$, be the smallest power of m such that it has the nonzero coefficient $c_s \leq k$. We can rewrite $c_s m^s$ as

$$c_s m^s = (c_s - 1)m^s + m^s \cdot m^0.$$

This gives us the partition

$$c_j m^j + \cdots + (c_s - 1)m^s + \cdots + c_1 m^1 + (c_0 + m^s)m^0.$$

Let $c_0^* = c_0 + m^s$.

If $c_0^* = k$, we're done.

If $c_0^* < k$, repeat from $(*)$ above.

If $c_0^* > k$, then we know that $c_0^* \equiv a (\text{mod } m)$ and $c_0^* - k \equiv 0 (\text{mod } m)$.

We will consider the $c_0^* - k$ extra copies of m^0 and combine them into groups of m to get $\frac{c_0^* - k}{m}$ new copies of m^1.

This provides the partition

$$c_j m^j + \cdots + (c_s - 1)m^s + \cdots + \left(c_1 + \frac{c_0^* - k}{m}\right) m^1 + km^0.$$

Let $c_1^* = c_1 + \frac{c_0^* - k}{m}$.

If $c_1^* \leq k$, we're done.

If we have $c_1^* > k$, then we will perform a similar regrouping of the m^1's.

Let $d_1 \leq k$ such that $c_1^* - d_1 \equiv 0 (\text{mod } m)$. Now combine those $c_1^* - d_1$ m^1's into groups of m to get $c_2^* = c_2 + \frac{c_1^* - d_1}{m}$. Now we have

$$c_j m^j + \cdots + (c_s - 1)m^s + \cdots + c_2^* m^2 + d_1 m^1 + km^0.$$

If necessary (i.e. $c_i^* > k$) continue to repeat the following regrouping for $2 \leq i < s$:

Let $d_i \leq k$ such that $c_i^* - d_i \equiv 0 (\text{mod } m)$. Now combine those $c_i^* - d_i$ m^i's into groups of m to get $c_{i+1}^* = c_{i+1} + \frac{c_i^* - d_i}{m}$.

When finished we will have a k^{th} m-ary partition of n that has exactly k ones. QED

To illustrate the algorithm in the proof above consider a 6^{th} 3-ary partition of 84 that we can transform into a partition that has exactly 6 ones:

$$
\begin{aligned}
84 &= 81 + 1 + 1 + 1 \\
&= 1 \cdot 3^4 + 3 \cdot 3^0 \\
&= 0 \cdot 3^4 + 84 \cdot 3^0 \\
&= 26 \cdot 3^1 + 6 \cdot 3^0 \\
&= 8 \cdot 3^2 + 2 \cdot 3^1 + 6 \cdot 3^0 \\
&= 2 \cdot 3^3 + 2 \cdot 3^2 + 2 \cdot 3^1 + 6 \cdot 3^0 \\
&= 27 + 27 + 9 + 9 + 3 + 3 + 1 + 1 + 1 + 1 + 1 + 1
\end{aligned}
$$

The Main Results

Summarizing the results from Lemmas 3.4 and 3.5 we have

Corollary 3.6.1. *If $n \not\equiv m - 1(\textit{mod } m)$, then*

$$b_m(n) = b_m(n + 1).$$

Proof. Let $n \not\equiv m - 1(\text{mod } m)$, and let $a = n \mod m$. From Lemmas 3.4 and 3.5, we have a bijection between the m-ary partitions of n and those of $n + 1$. Since adding 1 to the m-ary partitions of n yields the m-ary partitions of $n + 1$ (Lemma 3.4) and every m-ary partition of $n + 1$ has at least $a + 1$ ones, we can subtract 1 from each partition to get the m-ary partitions of n (Lemma 3.5). Thus

$$b_m(n) = b_m(n + 1).$$

QED

Now we can show that when we consider the additional restriction of each part only being allowed to appear at most k times, we get the result of Theorem 3.2 restated here below.

Theorem 3.2. *Let $m, n, k \in \mathbb{N}$ such that $1 \leq k \leq n$ and $m \geq 3$. If $k \equiv n(\text{mod } m)$ and $n \not\equiv m - 1(\text{mod } m)$, then*

$$b_m(k, n) > b_m(k, n + 1).$$

Proof of Theorem 3.2. The k^{th} m-ary partitions of n are a subset of size $b_m(k, n)$ of the m-ary partitions of n. By Lemma 3.4 we can append an extra addend of 1 onto each of these partitions to get an m-ary partition of $n + 1$. We group these new partitions into a set S, and notice this is a subset of size $b_m(k, n)$ of the set of m-ary partitions of $n + 1$.

Note that all of the partitions in S have at most k of each m^i-part where $i > 1$, and this is true for none of the m-ary partitions of $n + 1$ not included in S. Thus $b_m(k, n) \geq b_m(k, n + 1)$, because the k^{th} m-ary partitions of $n + 1$ must be a subset of S.

Now consider the number of $1'$s in each of the elements of S. Recall from Lemma 3.6 that at least one of the k^{th} m-ary partitions of n must have exactly k ones. We can write this partition as

$$n = \sum_{i=1}^{j} c_i m^i + k m^0 \quad \text{where } c_i \in \{0, 1, \ldots, k\}.$$

Then the partition of $n + 1$ associated with the partition above is

$$n + 1 = \sum_{i=1}^{j} c_i m^i + (k + 1) m^0 \quad \text{where } c_i \in \{0, 1, \ldots, k\}.$$

This partition is an element of S but cannot be a k^{th} m-ary partition of $n + 1$, since it has $k + 1$ ones.

Thus there is at least one fewer k^{th} m-ary partition of $n+1$ than there are of n. Hence $b_m(k, n) > b_m(k, n+1)$. QED

We can now also verify Theorem 3.3.

Theorem 3.3. *Let $m, n, k \in \mathbb{N}$ such that $1 \leq k \leq n$ and $m \geq 3$. If $k \equiv m-1(\text{mod } m)$ and $n \not\equiv m-1(\text{mod } m)$, then*

$$b_m(k, n) = b_m(k, n+1).$$

Proof of Theorem 3.3. We can represent the k^{th} m-ary partitions of n as

$$n = c_j m^j + c_{j-1} m^{j-1} + \cdots + c_1 m^1 + c_0 m^0.$$

By definition there are $b_m(k, n)$ such partitions.

Since $n \not\equiv m-1(\text{mod } m)$ we must have $c_0 \not\equiv m-1(\text{mod } m)$ and $c_0 < k$. Adding 1 to both sides of the partitions described above yields

$$n+1 = c_j m^j + c_{j-1} m^{j-1} + \cdots + c_1 m^1 + (c_0+1) m^0.$$

Now we have $c_0 + 1 \leq k$.

So all of the partitions above are k^{th} m-ary partitions of $n+1$. In fact, they are all $b_m(k, n+1)$ of the k^{th} m-ary partitions of $n+1$, since all possible values of c_i for $1 \leq i \leq j$ such that $c_i \leq k$ are already considered above. Thus regrouping the $c_0 + 1$ ones will give us an extra copy of a partition we have already counted. Also note that adding any other m^i to the partitions of n in lieu of m^0 will cause the resulting sum to be greater than $n+1$.
Hence $b_m(k, n) = b_m(k, n+1)$. QED

Connection to Base *m* Representation

Notice that in Equation (2.1) of $b_2(1,n)$ we have the number of ways of writing n in base 2, and in Theorem 3.1a we have the number of ways of writing n in base $m \geq 2$. Thus, we are guaranteed that there is exactly one way to do so for all n. The base m representation of n

$$n = c_j m^j + c_{j-1} m^{j-1} + \cdots + c_1 m^1 + c_0 m^0 \quad \text{where } c_i \in \{0, 1, \ldots, m-1\}$$

will be denoted by

$$n = (c_j c_{j-1} \ldots c_1 c_0)_m.$$

As we noted after the proof of Theorem 3.1a, this is the $(m-1)^{st}$ m-ary partition of n. Notice that in the $k = 1$ columns in Tables 3.1 and 3.2 there is an odd pattern of 0s and 1s. This is due to the restriction of using only one of each power of m, the pattern of 0s and 1s can be predicted as in the following theorem.

Theorem 3.7. *The partition function $b_m(1,n) = 1$ if and only if n can be written in base m using only the digits 0 or 1 in the representation. Otherwise $b_m(1,n) = 0$.*

For example, this pattern can be observed in Table 3.3.

Proof of Theorem 3.3. Assume $b_m(1,n) = 1$. Then there is a unique way to write n as distinct powers of m, namely:

$$n = c_j m^j + c_{j-1} m^{j-1} + \cdots + c_1 m^1 + c_0 m^0 \ \text{ for } c_i \in \{0, 1\}.$$

TABLE 3.3
Base m Representation of n.

n	base 3 representation	$b_3(1,n)$	base 4 representation	$b_4(1,n)$
0	$(000)_3$	1	$(000)_4$	1
1	$(001)_3$	1	$(001)_4$	1
2	$(002)_3$	0	$(002)_4$	0
3	$(010)_3$	1	$(003)_4$	0
4	$(011)_3$	1	$(010)_4$	1
5	$(012)_3$	0	$(011)_4$	1
6	$(020)_3$	0	$(012)_4$	0
7	$(021)_3$	0	$(013)_4$	0
8	$(022)_3$	0	$(020)_4$	0
9	$(100)_3$	1	$(021)_4$	0
10	$(101)_3$	1	$(022)_4$	0
11	$(102)_3$	0	$(023)_4$	0
12	$(110)_3$	1	$(030)_4$	0
13	$(111)_3$	1	$(031)_4$	0
14	$(112)_3$	0	$(032)_4$	0
15	$(120)_3$	0	$(033)_4$	0

Then we have that the base m representation of n is

$$n = (c_j c_{j-1} \dots c_1 c_0)_m$$

where the digits c_i are the same as in the partition above, either 0 or 1.

Suppose n can be written in base m only using the digits 0 or 1. Then

$$\begin{aligned} n &= (c_j c_{j-1} \dots c_1 c_0)_m \qquad \text{where } c_i \in \{0,1\} \\ &= c_j m^j + c_{j-1} m^{j-1} + \cdots + c_1 m^1 + c_0 m^0. \end{aligned}$$

This is an m-ary partition of n, where each power of m is used at most once (i.e. a 1^{st} m-ary partition), so there are $b_m(1,n)$ of them. It is well know that each positive integer has a unique representation in base m [2]. So $b_m(1,n) = 1$.

Thus far we have verified that the partition function $b_m(1,n) = 1$ if and only if n can be written in base m using only the digits 0 or 1 in the representation. Now we will show $b_m(1,n) = 0$ if and only if n cannot be written in base m using only the digits 0 or 1 in the representation.

If $b_m(1,n) = 0$, then there does not exist a way to write n as a sum of distinct powers of m. Thus we have at least one c_i in the m-ary partition of n such that $1 < c_i \leq m-1$. Note that this c_i cannot be greater than $m-1$, since this might allow the possibility of grouping or separating the $c_i m^i$ into distinct powers of m. So this gives the base m representation

$$n = (c_j c_{j-1} \ldots c_i \ldots c_1 c_0)_m$$

that is guaranteed to have at least one digit (c_i) that is not 0 or 1.

Lastly assume

$$n = (c_j c_{j-1} \ldots c_1 c_0)_m$$

where there must be at least one c_i such that $1 < c_i \leq m-1$. We know this unique representation of n can also be written as

$$n = c_j m^j + c_{j-1} m^{j-1} + \cdots + c_i m_i + \cdots + c_1 m^1 + c_0 m^0$$

where the m^i part is not distinct.

Thus this is not a 1^{st} m-ary partition of n, so $b_m(1,n) = 0$. QED

CHAPTER 4

FURTHER QUESTIONS

From our observations of $b_3(k,n)$ and $b_4(k,n)$, specifically Equation (3.2), we proved Theorem 3.3. However, this theorem neglects the case when $n \equiv m-1 (\text{mod } m)$. In this case we will have $b_m(k,n) \leq b_m(k,n+1)$, if $k < m^2-1$; and $b_m(k,n) < b_m(k,n+1)$, if $k \geq m^2-1$.

Conjecture. *If $k \equiv m-1 (\text{mod } m)$, then $b_m(k,n) \leq b_m(k,n+1)$ for all n.*

For example, consider the 5^{th} 3-ary partitions for the given n and $n+1$. We ordered the partitions of $n+1$ such that a partition that results from adding a part of 1 to a partition of n is written across from that partition of n. An "X" signifies that adding 1 does not yield a 5^{th} 3-ary partition of $n+1$.

$n=2$	$n+1=3$
1+1	1+1+1
	3

$n=5$	$n+1=6$
3+1+1	3+1+1+1
1+1+1+1+1	X
	3+3

$n=8$	$n+1=9$
3+3+1+1	3+3+1+1+1
3+1+1+1+1+1	X
	9
	3+3+3+3

$n = 11$	$n + 1 = 12$
9+1+1	9+1+1+1
3+3+3+1+1	3+3+3+1+1+1
3+3+1+1+1+1+1	X
	9+3
	3+3+3+3

$n = 14$	$n + 1 = 15$
9+3+1+1	9+3+1+1+1
9+1+1+1+1+1	X
3+3+3+3+1+1	3+3+3+3+1+1+1
3+3+3+1+1+1+1+1	X
	9+3+3
	3+3+3+3+3

Notice that in each table there are extra partitions of $n+1$ not formed by appending a 1 onto the partitions of n. If we are able to show that the number of partitions of $n+1$ that are not obtained from simply adding 1 to a partition of n is greater than or equal to the number of partitions of n with exactly k ones, then we can prove the conjecture.

It would also be interesting to explore the monotonicity properties of Rødseth and Sellers' partition function $b_{m,k}(n)$ [20] as mentioned in Section 2.4, where we have

$$n = \sum_{i \geq 0} c_i m^i \text{ where } c_i \in \{0, 1, 2, \ldots, m^k - 1\}.$$

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APPENDIX A

Mathematica Code

For Table 1.1, to find the first 100 terms of the sequence of $p(n)$ we use the generating function of $p(n)$, $g(x) = \prod_{i\geq 0} \frac{1}{1-x^i}$:

```
CoefficientList[
 Series[Product[((1)/(1 - x^(i))), {i, 1, 100}], {x, 0, 99}], x]
```

This will be accurate to $p(100)$.

For the values of $b_2(n)$ in Table 2.1, we will use the generating function $\prod_{i\geq 0} \frac{1}{1-x^{2^i}}$. With $0 \leq i \leq 6$ we will have values accurate for all $n \leq 127$.

```
CoefficientList[
 Series[Product[((1)/(1 - x^(2^i))), {i, 0,6}], {x, 0, 99}], x]
```

For the first of the Churchhouse conjectures

$$b_2(2^{2k+2}t) - b_2(2^{2k}t) \equiv 0 (\text{mod } 2^{3k+2})$$

we can use Table 2.2 to give some examples of the above expression. These values are found by using the code:

```
For[k = 1, k < 5, k++,
 For[t = 1, t < 6, t = t + 2,
  Print[k, "--", t, "--",
   SeriesCoefficient[
     Series[Product[((1)/(1 - x^(2^i))), {i, 0, 12}], {x, 0,
       25000}], (2^(2 k + 2)*t)] -
    SeriesCoefficient[
      Series[Product[((1)/(1 - x^(2^i))), {i, 0, 12}], {x, 0,
```

```
        25000}], (2^(2 k)*t)]
   ]
  ]
 ]
```

This gives us $b_2(2^{2k+2}t) - b_2(2^{2k}t)$ for $1 \leq k \leq 4$ and odd $t \leq 5$. For the values of i, $0 \leq i \leq 12$ allows for accuracy up to $b_2(8191)$.

In Table 2.3 we illustrate the second of the Churchhouse conjectures,

$$b_2(2^{2k+1}t) - b_2(2^{2k-1}t) \equiv 0 (\text{mod } 2^{3k})$$

Similar to the code above we have:

```
For[k = 1, k < 5, k++,
 For[t = 1, t < 6, t = t + 2,
  Print[k, "--", t, "--",
   SeriesCoefficient[
     Series[Product[((1)/(1 - x^(2^i))), {i, 0, 12}], {x, 0,
       11000}], (2^(2 k + 1)*t)] -
    SeriesCoefficient[
     Series[Product[((1)/(1 - x^(2^i))), {i, 0, 12}], {x, 0,
       11000}], (2^(2 k - 1)*t)]
   ]
  ]
 ]
```

In Table 2.4 the values of $b_m(n)$ for $3 \leq m \leq 10$ are found by using the generating function $B_m(x) = \prod_{i \geq 0} \frac{1}{1-x^{m^i}}$:

```
For[m = 3, m < 11, m++,
Print[m, "--", CoefficientList[
  Series[Product[((1)/(1 - x^(m^i))), {i, 0, 3}], {x, 0, 30}], x]
 ]
]
```

Where $0 \leq i \leq 3$ makes this accurate for all $n \leq m^4 - 1$.

In Theorem 2.3, Andrews gives a generalization of the Churchhouse conjectures

$$b_m(m^{r+1}n) - b_m(m^r n) \equiv 0 (\text{mod } \mu^r)$$

where $\mu = m$ if m is odd and $\mu = \frac{m}{2}$ if m is even. Table 2.5 porvides values for $b_3(3^r n)$ to illustrate the theorem. We obtain the values in that table from:

```
For[i = 1, i < 6, i++,
  For[ r = 1, r < 6, r++,
   Print[i, "--", r, "--",
   SeriesCoefficient[
    Series[Product[1/(1 - x^(3^i)), {i, 0, 10}], {x, 0, 1500}],
    3^(r)*i]
    ]
  ]
 ]
```

In Table 2.6 we can use the generating function $H_m(x) = \prod_{i\geq 0} \frac{1-x^{(m+1)m^i}}{1-x^{m^i}}$ to find the values of the hyper-m-ary partition function, $h_m(n)$, for $2 \leq m \leq 10$:

```
For[m = 2, m < 11, m++,
Print[m, "--", CoefficientList[
  Series[
   Product[((1-x^((m+1)*m^i))/(1 - x^(m^i))), {i, 0, 10}],
  {x, 0, 30}], x]
 ]
]
```

For $b_2(k,n)$ in Table 2.7 we can find the values for $1 \leq k \leq 10$ using the generating function $G_{2,k}(x) = \prod_{i\geq 0} \frac{1-x^{(k+1)2^i}}{1-x^{2^i}}$ we have:

```
 For[k = 1, k < 11, k++,
Print[k, "--", CoefficientList[
   Series[Product[((1-x^((k+1)*2^i))/(1 - x^(2^i))), {i, 0, 10}],
  {x, 0, 30}], x]
  ]
 ]
```

Similarly for $b_3(k,n)$ and $b_4(k,n)$ in Tables 3.1 and 3.2 by replacing the 2^is with 3^i and 4^i, respectively.